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9. Multi-parameter fracture mechanics: Practical use

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9.1. Introduction

Classical linear elastic fracture mechanics is certainly the most common tool for assessment of fracture response/behaviour of various structures/materials. It considers the stress intensity factor (SIF) as the single parameter expressing the amplitude of the near-crack tip stress field and controlling the (un)stable crack growth. This well-known theory is derived for brittle materials and is rather sufficiently described. Nevertheless, there exist materials where the fracture process occurs not only at the crack tip (and in its very vicinity) but it takes place in a large zone of structure ahead of the crack tip. The material behaviour in this region is mostly nonlinear and is not explained and described adequately. Moreover, if the zone with nonlinear behaviour is of a large extent (comparable to the structural dimensions), the size/geometry/boundary effect cannot be omitted, see (Ayatollahi and Akbardoost 2012, Duan et al. 2007 or Karihaloo et al. 2006).

It has been shown that the so-called multi-parameter approach for approximation of the crack-tip stress and displacement fields can help to express the stress distribution better than if the only one or two parameters are used (as it is usual), see e.g. (Šestáková 2014, Veselý et al. 2014) for more details. Further investigations in this area are introduced in this paper. The Williams expansion (WE), see (Williams 1957), is used for the stress state description and subsequently also as an input for the fracture criteria enabling prediction of the crack path and estimation of the plastic zone size. It means that the criteria are used in a kind of their generalized form. This study represents a short summary of the authors' complex investigations devoted to influence of the higher-order terms of the WE on various fracture characteristics/phenomena (near-crack-tip stress field approximation, crack propagation angle, plastic zone extent, etc.), see for instance the works cited above in this paragraph or ((Šestáková) Malíková 2013a) for more details.

9.2. Theoretical background

The main idea of this paper is based on the theory derived by Williams (Williams 1957). He showed that the stress and displacement fields around the crack tip in an elastic isotropic homogeneous body subjected to an arbitrary remote loading can be expressed in a form of a series expansion, particularly as a power series. Thus, the stress tensor and displacement vector components for a cracked plate loaded in a combination of the I and II loading mode can be written as:

$$\sigma_{ij} = \sum_{n=1}^{\infty} A_n \frac{n}{2} r^{\frac{n}{2}-1}\, {}^{\sigma}f_{\mathrm{I},ij}(\theta, n) + \sum_{m=1}^{\infty} B_m \frac{m}{2} r^{\frac{m}{2}-1}\, {}^{\sigma}f_{\mathrm{II},ij}(\theta, m) \quad (9.1)$$

and

$$u_i = \sum_{n=0}^{\infty} A_n \frac{n}{2} r^{\frac{n}{2}}\, {}^{u}f_{\mathrm{I},i}(\theta, n, E, \nu) + \sum_{m=0}^{\infty} B_m \frac{m}{2} r^{\frac{m}{2}}\, {}^{u}f_{\mathrm{II},i}(\theta, m, E, \nu) \quad (9.2)$$

Where:

i, j	–	$i, j = \{x, y\}$,
r, θ	–	polar coordinates (assuming the origin of the coordinate system at the crack tip and the crack faces lying on the negative x-axis)
${}^{\sigma}f_{I,ij}(\theta, n)$	–	known functions which can be found in various fundamental works on fracture mechanics and related papers (Williams 1957, Ayatollahi and Nejati 2011, Anderson 2004)
${}^{\sigma}f_{II.ij}(\theta, m)$	–	ditto
${}^{u}f_{I.i}(\theta, n, E, \nu)$	–	ditto
${}^{u}f_{\mathrm{II}.i}(\theta, m, E, \nu)$	–	ditto
E, ν	–	material parameters, i.e. Young's modulus and Poisson's ratio
A_n, B_m	–	depend on the specimen geometry and boundary conditions; they are constant for a particular crack propagation configuration and for a particular relative crack length $\alpha = a/W$, respectively (a is the crack length and W is the specimen width).

It should be mentioned that the first (singular) term coefficients, A_1 and B_1, in Eq. 9.1 are related to the well-known mode I and mode II stress intensity factors K_I and K_II, respectively ($K_\mathrm{I} = A_1\sqrt{2\pi}$, $K_\mathrm{II} = -B_1\sqrt{2\pi}$). These parameters are dominant for $r \to 0$, which is the main idea of the conventional linear elastic fracture mechanics approach. It means that in materials where brittle fracture occurs and where the extent of the failure zone around the crack tip is negligibly small, the stress intensity factor can be used as the single-controlling parameter for failure assessment.

Coefficients of the higher-order terms (n, $m > 1$) were somewhat ignored in the past and are generally not connected to any conventional fracture parameters; except for the second term that corresponds to the in-plane T-stress. Generally, this parameter can contribute to a more accurate utilization of the fracture resistance value obtained from measurements on laboratory size specimens within the fracture response assessment of a real structure (applied mainly in the case of brittle fracture of metals, ceramics etc.). However, it has been shown that other higher-order terms can also possess an essential importance, especially if the non-linear zone extent around the crack tip is very large in relation to the structure dimensions, which occurs in elastic-plastic and quasi-brittle materials. In these cases, the higher-order terms of the WE should be taken into account for description of the stress state near the crack tip in order to describe it with sufficient precision in the entire region, where the quasi-brittle fracture response or(and) the crack-tip plasticity takes place. Thus, when more than two terms of the Williams asymptotic expansion are considered, an approach applying the so-called multi-parameter fracture mechanics is referred to.

9.3. Over-deterministic method

Numerical techniques (such as the finite element method, FEM) for estimation of the coefficients of the WE are unavoidable in most engineering tasks. Moreover, the coefficients of the third and the higher-order terms of the crack tip asymptotic field are not easy (and common) to obtain. So far, there have been only a few FE techniques introduced, enabling calculation of the coefficients of the higher-order terms. For instance, (Karihaloo and Xiao 2001) have improved and extended the hybrid crack element (HCE) introduced by (Tong et al.1997). (Su and Fok 2007) applied the fractal FE method (FFEM) together with 9-node Lagrangian hybrid elements to calculate the coefficients of the WE. Also the boundary collocation method (BCM) can be used (Xiao et al. 2004). However, all the methods mentioned above use advanced mathematical procedures and more extensive and deeper knowledge of the special elements or FE code is unavoidable. Therefore, the so-called over-deterministic approach (ODM) based on the formulation of linear least-squares is employed in this paper (well described for instance in (Ayatollahi and Nejati 2011). The main advantage of this method is that conventional finite elements can be used for the solution and only knowledge of the displacement field near the crack tip is required. Note that a more general approach of the eigenvector solution is presented in work by (Carpenter 1985).

The application of the ODM is based on Eq. 9.2, which is crucial and requires displacement field data as inputs. Thus, the evaluation procedure starts with a standard numerical simulation of a cracked specimen where a set of nodes (usually from a vicinity of the crack tip) is chosen and their coordinates and displacements are used as inputs for Eq. 9.2. In particular, two components, u and v, of the displacement vector exist in each node (in a 2-dimensional task) and therefore two equations can be written. Consequently, a system of $2k$ algebraic equations for variables A_n and B_m exists, where k represents the number of nodes selected around the crack tip for the ODM application. In order to satisfy the principle of the method (based on an over-determined system of equations), a relation between the number of nodes selected around the crack tip k and the numbers of terms of the power series considered for the stress state approximation N, M has to follow the inequality $2k > N + M + 2$.

It generally holds that the number of the higher-order terms N, M can be chosen arbitrarily, but there exist some empirical restrictions/recommendations with regard to improving the results, see for instance (Ayatollahi and Nejati 2011). The authors of the present paper have also devoted several publications to parametric studies on the ODM accuracy, convergence, or mesh sensitivity etc. (Šestáková (Malíková) 2013b).

9.4. Computational tools

Ordinary FEM. It has been mentioned above that a conventional FE analysis is sufficient for application of the over-deterministic method. Therefore, a standard finite elements model of the cracked configurations under investigation was created and analysed in Ansys FE software (ANSYS 2005). 2D 8-node iso-parametric elements with quadratic basis functions (labeled as PLANE82 in the Ansys FE system) were used for modelling of the specimen. The existence of the mid-side nodes (KSCON) in the elements used for modelling of the crack tip in the specimens is very important for the KCALC built-in procedure for evaluation of stress intensity factor value. The crack-tip singularity in the stress field is enabled to be simulated/described by means of shifting of the mid-side nodes towards the crack tip.

Mathematica software. The procedure for calculation of the Williams expansion coefficients (via ODM) was programmed in Mathematica software (Wolfram Mathematica 2007). The same code has been used for analytical reconstruction of the stress field, which resulted in the creation of contour plots of stress distribution in the cracked specimens (Veselý et al. 2014).

9.5. Fracture mechanics tasks under investigation

9.5.1. Stress field reconstruction

The values of the higher-order terms coefficients for a selected mixed-mode geometry (Angled Edge-Crack under uniaxial Tension, AECT, see Fig. 9.1) were determined for a wide range of relative crack lengths in order to obtain suitable polynomial approximations by means of fitting the data calculated. These formulas can then be used for reconstruction of the stress/displacement field for an arbitrary crack length. In Fig. 9.2 the stress fields obtained from the Williams approximation are compared with results calculated numerically for the relative crack length $\alpha = 0.8$; note that Rankine failure criterion is considered.

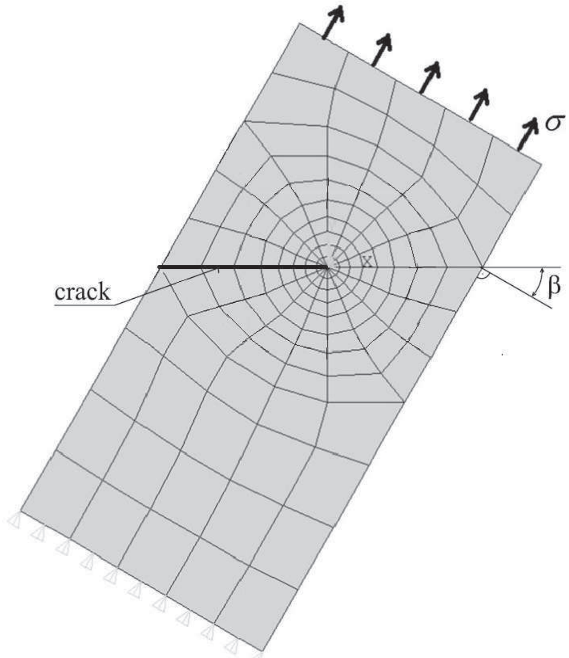

Fig. 9.1. Schema of the cracked mixed-mode geometry under investigations

It is observed that the lower stress levels and/or stress fields farther from the crack tip should be reconstructed by means of more than only one or two parameters (as it is common for brittle materials where the singular stress is dominant because the fracture occurs only in a small region very close to the crack tip). The so-called multi-parameter fracture mechanics approach presented in this work is more general and can be applied for larger range of materials with various (non-linear) fracture behaviour. Unfortunately, the sufficient number of higher-order terms necessary for accurate stress and displacement field description within a body with a crack depends on the size of the region in question and cannot be generalized.

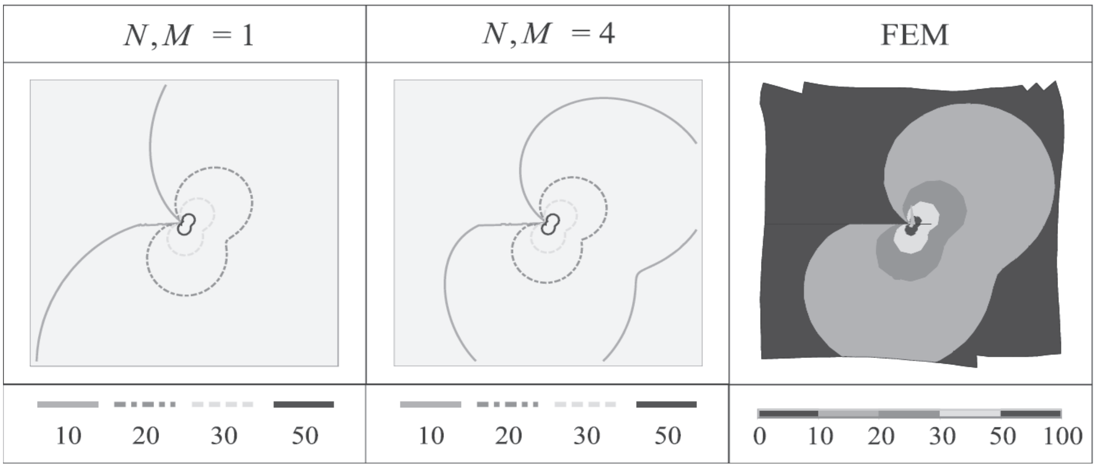

Fig. 9.2. Comparison of σ_1 principal stress field in the AECT specimen with the relative crack length $\alpha = 0.8$ reconstructed by means of the WE using various numbers of terms of the power series (1 and 4, respectively) with the numerical solution obtained directly from FEM

Note that the study performed is of larger extent, see (Malíková and Veselý 2015) for more details. It can be seen that the higher-order terms of the WE can be important. According to the study presented, the multi-parameter fracture mechanics approach is necessary especially when the stress field is influenced by the boundary conditions (short/long cracks) and/or when the stress field is investigated at larger distances from the crack tip. Accurate stress distribution is essential for further fracture analyses, such as estimation of the fracture process zone size (and its relation to other fracture effects), crack propagation direction (Shahani 2009, Ševčík et al. 2013) fracture toughness (Sahgafi et al. 2010), size effects (Ayatollahi and Akbardoost 2012, Ayatollahi and Akbardoost 2014), etc.

9.5.2. Crack path investigation

An anti-symmetrical four-point bending specimen (EA4PB), see Fig. 9.3, has been chosen for investigation of the crack propagation and the maximum tangential stress (MTS) criterion applied in order to find the kink angle γ.

The criterion was used in its well-known explicit form (Sih and Erdogan 1963 or Qian and Fatemi 1996), see Eq. 9.3, as well as in its generalized form, see Eq. 9.4, considering the $\sigma_{\theta\theta}$ approximation via WE taking into account various numbers of the initial terms of the series:

$$\gamma = 2\arctan\frac{-2K_{\mathrm{II}}}{K_{\mathrm{I}} + \sqrt{K_{\mathrm{I}}^2 + 8K_{\mathrm{II}}^2}} \tag{9.3}$$

$$\frac{\partial \sigma_{\theta\theta}}{\partial \theta} = 0 \quad , \quad \frac{\partial^2 \sigma_{\theta\theta}}{\partial \theta^2} < 0 \tag{9.4}$$

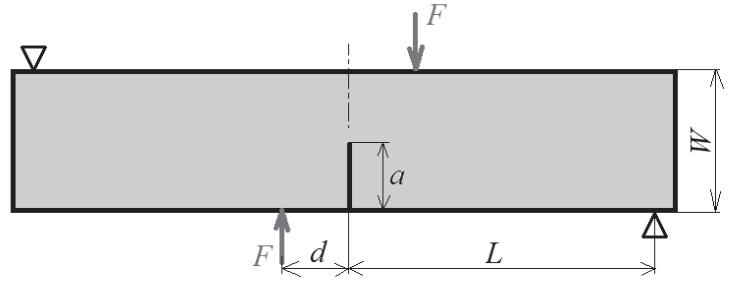

Fig. 9.3. Schema of the cracked anti-symmetrical four-point bending specimen

If the crack path through a specimen shall be estimated by means of the MTS criterion, knowledge of the tangential stress distribution is essential. In Fig. 9.4 the deformed anti-symmetrical four-point bending specimen with a propagating crack is plotted; $\sigma_{\theta\theta}$ contour lines in the most important region near the crack tip are highlighted.

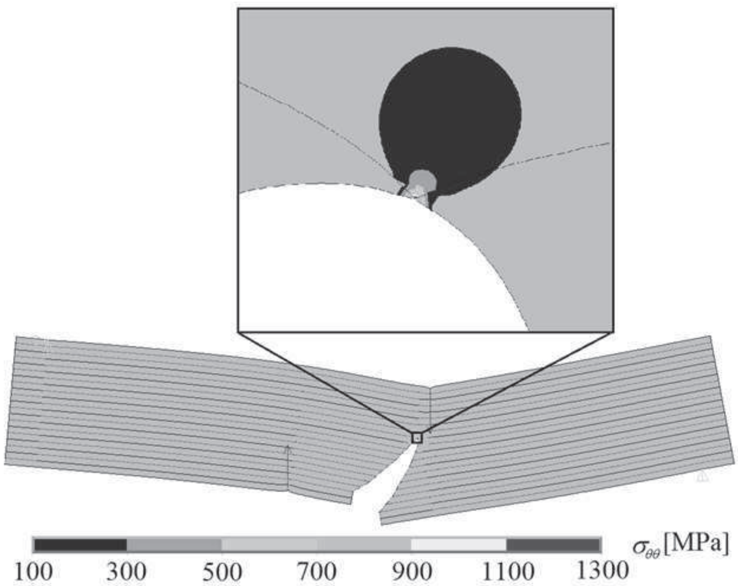

Fig. 9.4. Tangential stress contour lines in the deformed (200 times enlarged) anti-symmetrical four-point bending specimen with a propagating edge crack with its tip in a depth of 24 mm in the specimen

Several first steps of the crack path through the specimen investigated are presented in Fig. 9.5. The crack paths have been calculated by means of both the classical MTS criterion (when only the first singular term of the expansion is taking into account) and the generalized form of the MTS criterion (when the initial 2, 3, 4 or 5 terms of the WE are taking into account). Both criteria have been applied on various radii.

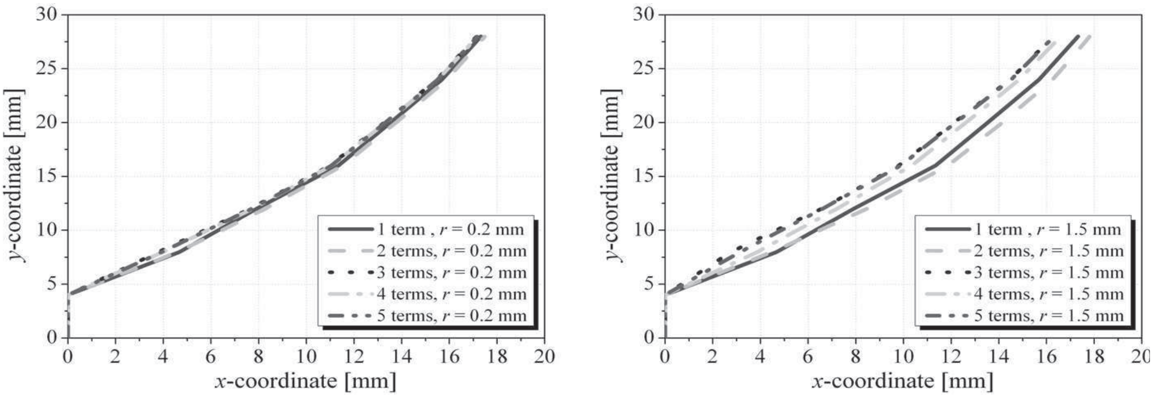

Fig. 9.5. Crack paths through the specimen calculated via generalized MTS criterion taking into account various numbers of terms of the WE and applied on the radius of 0.2 mm (left) and 1.5 mm (right) around the crack tip

In this study, see more details in ((Šestáková) Malíková 2013a), the Williams expansion derived for description of stress/displacement crack-tip fields is introduced as a tool for generalization of the classical MTS criterion used for determination of the crack propagation direction in an anti-symmetrical four-point bending specimen. The study presented shows that if the distance from the crack tip, where the criterion shall be applied, is large enough, using the multi-parameter fracture mechanics approach, i.e. the generalized MTS criterion, gives more accurate estimation of the crack path. Application of the classical MTS criterion in this case could decrease the accuracy because it predicts that the crack does not propagate so rapidly as the crack path estimation from the generalized MTS criterion shows.

9.5.3. Plastic zone extent estimation

The same mixed mode configuration as in the previous section, see Fig. 9.3, was chosen for the presented study because of its generality in comparison to specimens loaded in pure mode I or pure mode II. In this analysis, the plastic zone extent is investigated by means of the Rankine criterion for the selected EA4PB mixed-mode configuration. Particularly, the influence of the number of the WE terms considered for the stress components approximation on the plastic zone size estimation is studied. In Fig. 9.6 comparison of the plastic zone size calculated via FEM with results calculated from the same criterion by means of the WE with various numbers of the initial terms. Three different cases are simulated:

materials with the critical stress/tensile strength value $\sigma_c = 4$, 6 and 8 MPa are assumed. Note that the dimensions of the plotted area are 15.4 mm × 15.4 mm.

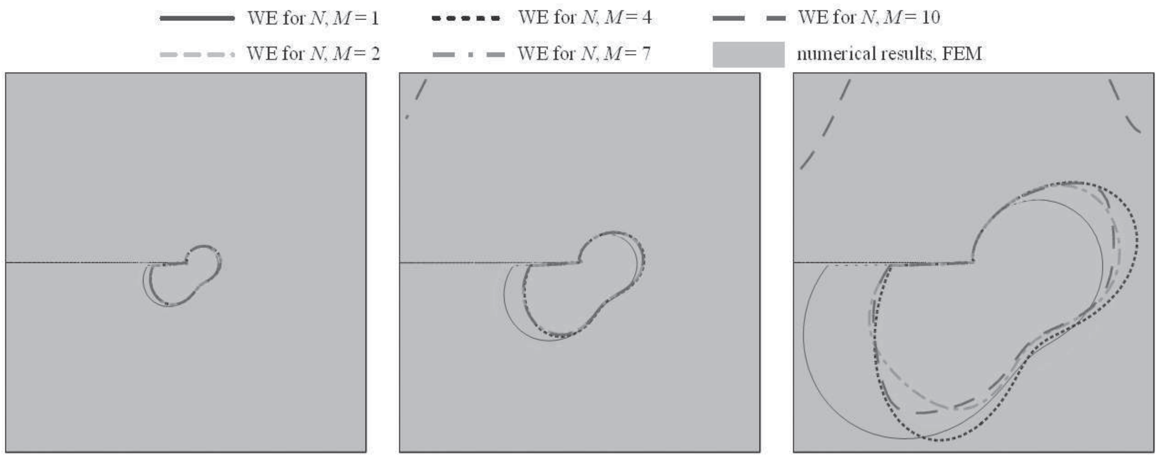

Fig. 9.6. Comparison of the plastic zone sizes calculated via FEM using Rankine criterion for the chosen EA4PB configurations with results calculated from the same criterion when the stress components are approximated via the WE with various numbers of the initial terms and considering the critical stress value: $\sigma_c = 8$ MPa (left); $\sigma_c = 6$ MPa (middle); $\sigma_c = 4$ MPa (right)

The plastic zone extent was investigated in an eccentric asymmetric four point bending specimen using the Rankine/maximum stress criterion, see more details in (Malíková and Veselý 2014). By means of varying the tensile strength of the material in the numerical model of the cracked specimen, it is shown that when the nonlinear zone extent is small enough, the classical one- or two-parameter fracture mechanics concept can be applied. On the other hand, when the stress field needs to be investigated also in larger distances from the crack tip, i.e. the plastic zone extent is comparable to the typical specimen dimensions, using the WE with more higher-order terms for the stress field approximation can be very advantageous. Then, a generalized form of the criterion is spoken about and its utilization can contribute to better description of the occurring failure process.

9.6. Conclusions

This chapter should represent a short summary of the authors' complex investigations devoted to influence of the higher-order terms of the WE on various fracture characteristics/phenomena (near-crack-tip stress field approximation, crack propagation angle, plastic zone extent, etc.). It is shown that the higher-order terms can play a significant role for some specific test configurations and can help to increase the accuracy of assessment of fracture mechanics tasks.

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