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6. Plate strip in a stabilized temperature field and creep effect

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Abstract: Deflection and stress states of a plate strip with variable stiffness and Boussinesq half-space subjected to actions of discontinuous temperature field under assumption of plane deformation are investigated. The hereditary creep of the plate and of the subgrade materials according to different constitutive equations is applied. The formulation and the solution of the basic integro-differential equation system are performed in Laplace′s transforms using an undirect way. The time dependent solution is constructed by means of modified Erdélyi Schapery algorithm. The numerical method with a detailed analysis of the obtained results is demonstrated on a numerical example.

Keywords: plate strip, stress states, temperature, creep

6.1. Introduction

The temperature stresses and deformations can play significant, sometimes even prevailing part in the design of foundation plates and structures. It is concerned the road and airdrome plates, the foundation plates in metallurgical and chemical plants, foundation plates and blocks of dams etc. The magnitude of the temperature stresses and deformations can depend on the thermo-physical properties of concrete, on the temperature of fresh concrete, on the exothermic reaction of cement, on the temperature of ambient environment, i.e. on that of air, water, and subgrade, and on the interaction mode of the foundation structure and the subgrade. In evolution of temperature stresses and deformations, the relaxation properties of material of the foundation structures and of the subgrade play a non negligible part. As will be shown further, while e.g. the deflections of the plates are getting larger, the bending moments and the shear forces are getting smaller. These findings can have a great importance from the viewpoint of the life time of the foundation structures. Only a sporadic attention has been paid in literature to the time dependent problems of temperature and deformations in massive concrete blocks and plates. Among the former sources the works of the Armenian School represented by N. Ch. Arutjunjan and his collaborators (Arutjunjan, 1955), (Zadajan, 1957), and those of the Polish school headed by W. Nowacki (Nowacki, 1963) are to be mentioned. Nowadays, particular attention is paid to contact problems (Moravkova, 2017), (Farhatnia, 2017), (Korotchenko, 2017) but time-dependent deformations less (Mistrikova, 2007), (Mistrikova, 2012), (Sumec, 2010). This chapter deals with the temperature stresses and deformations of the plate strip with the rigid borders, which is freely supported on the half-space, due to temperature actions of the ambient environment. By means of stiff borders is simulated in solution the lateral concrete walls the stiffness of which can be considered as infinitely large with regard to the finite thickness of plate. For example, craft locks, sludge-digestion tanks, blocks of dams, and other. It is assumed that the ambient temperature is stabilized. The investigations is limited to the temperatures for which the thermal and material characteristics of the plate can be consider as the permanent magnitudes independent on the temperature. It is further contemplated that the materials of the plate and of the subgrade undergo the creep with time and that they fulfil the basic equations of the theory of hereditary creep proposed by L. Boltzmann & V. Volterra (Boltzmann, 1970), (Volterra, 1913). That theory is based on the assumption of linear relationship between stresses and strains for which the law of superposition is valid. Experimental measurements show that the both asumptions are very well fulfilled till to the 40 percent of strength of materials. Constitutive equations of the hereditary creep are the most satisfying for the rheological properties of concrete structures and of the real subgrades in the time of their exploitation.

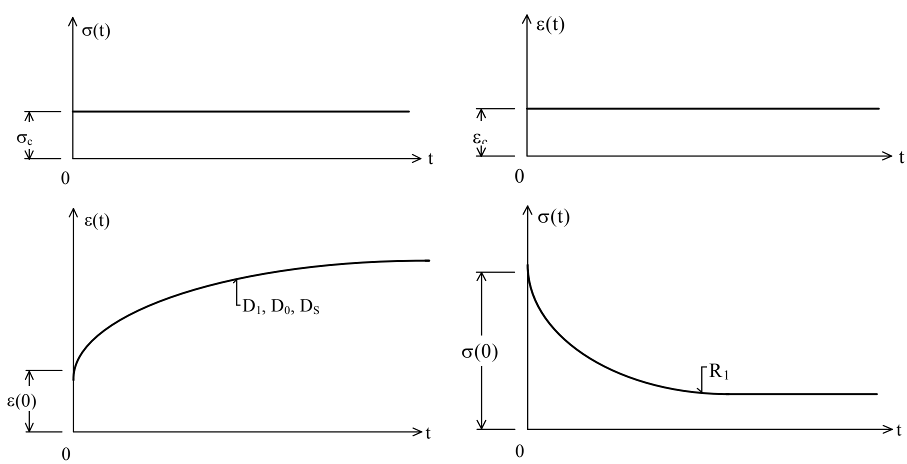


Fig. 6.1. Transient functions of creep D1 and those of relaxation R1 at uniaxial deformation

As the phenomena of creep and relaxation, i.e. the transient functions of creep and those of relaxation (Fig. 6.1) represent discontinuous functions of time, these physical variables are interpreted as so called Schwartz distributions (some type of the generalized functions with the carrier on the time half-axis <0, ∞). Such an interpretation permits to write down the constitutive equations in terms of the time invariant linear theory of the hereditary creep in a very compact form, as the equations with convolutions. Constructing the solution starting from the long term measurements for concrete (Loom and Base, 1990), (Neville, 1970) and for subgrade (Mesčjan, 1967), (Zareckij, 1993) it is assumed that concrete (subscript d) and subgrade (subscript z) are represented by quasielastic bodies having the instantaneous and the time variables values of Poisson′s ratio practically constant:

$$\mu_d(0) = \mu_d(t) = \text{const},\ \mu_z(0) = \mu_z(t) = \text{const}.$$

The basic equations are solved in Laplace′s transforms.The time dependent solution is constructed using Erdélyi – Scahapery method of inverse transformation modified by B. Novotný (Novotný, 1980). The formulation of the basic equations and their solution is represented using the dimensionless variable ξ given by the quocient of the real ordinate x and of a half of the foundation joint $\ell/2$, i.e. $\xi = 2x/\ell$ and then $\alpha = 2a/\ell$, $\beta = 2b/\ell$.

6.2 Basic equations and their solution

Assume that a plate strip with variable thickness is freely placed on a viscoelastic half-space and has a constant thickness h within the interval $< -\beta, +\beta >$, and that its stiffness is infinite within the extreme intervals $< -1, -\beta >$ and $< +\beta, +1 >$. It is assumed that the plate strip on a finite area $< -\alpha, \alpha >$ is subjected to a constant temperature $T(z)\,H(t)$, where $H(t)$ is Heaviside′s function, and that $T(z) = 0$ except this area. The thermal gradient is assumed to be variable along the thickness of the plate (Fig. 6.2).

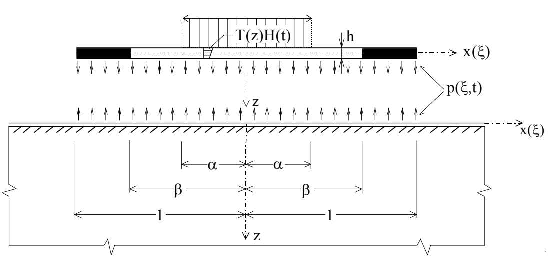


Fig. 6.2. Geometric scheme of the plate strip interacting with the half-space in a discontinuous temperature field *T(z)H(t)*.

The plate strip will be deformed due to influence of temperature. The stresses $p(\xi,t)$ will arise on the contact of the strip with the half-space. The problem is to determine the magnitude and the distribution way of these stresses within the time $t\in<0,\infty)$. The stresses $p(\xi,t)$ is determined from the equilibrium equations of the plate strip as a stiff entity

$$\int_{-1}^{+1} p(\xi,t)\,d\xi = 0, \quad \int_{-1}^{+1} \xi p(\xi,t)\,d\xi = 0 \qquad (6.1)$$

and from the deformation condition

$$w(\xi,t) = -v_z(\xi,t) \qquad (6.2)$$

expressing the equality of the plate strip deflections $w(\xi,t)$ and of the vertical displacements of the border ($z = 0$) of the half-space $v_z(\xi,t)$ in each point ξ of the interval $< -1, +1 >$ of the foundation joint. Deriving the equations for the deflections of the plate strip and for its internal forces is started from the constitutive equations in which the deformations of each element of plate are composed of the temperature deformation and of the viscoelastic deformation caused by temperature stresses.

$$\begin{aligned}
\sigma_x(\xi,t) &= \frac{1}{1-\mu_d^2} R_{d1}^{[1]} * \left[\varepsilon_x(\xi,t) - (1+\mu_d)\alpha_T T(z) H(t)\right] \\
\sigma_y(\xi,t) &= \frac{1}{1-\mu_d^2} R_{d1}^{[1]} * \left[\mu_d \varepsilon_x(\xi,t) - (1+\mu_d)\alpha_T T(z) H(t)\right] \qquad (6.3) \\
\tau_{xy}(\xi,t) &= 0
\end{aligned}$$

The relationships (6.3) are obtained from the following adjoint equations of Duhamel –Neumann (Duhamel, 1838)

$$\begin{aligned}
\sigma_x(\xi) &= \frac{E_d}{1-\mu_d^2} \left[\varepsilon_x(\xi) - (1+\mu_d)\alpha_T T(z)\right] \\
\sigma_y(\xi) &= \frac{E_d}{1-\mu_d^2} \left[\mu_d \varepsilon_x(\xi) - (1+\mu_d)\alpha_T T(z)\right] \qquad (6.4) \\
\tau_{xy}(\xi) &= 0
\end{aligned}$$

by assignment of the correspondent viscoelastic equivalent (Kovařík, 1987). In equations (6.3) the strain $\varepsilon_x(\xi,t)$ be means of the deflections $w(\xi,t)$ of the middle plane of the plate stripis expressed as follows:

$$\varepsilon_x\left(\xi,t\right)=-\frac{z}{\left(\frac{\ell}{2}\right)^2}\frac{\partial^2 w\left(\xi,t\right)}{\partial\xi^2} \tag{6.5}$$

Then the stress resultants can be expressed, i.e. the bending moment and the shear forces in the following form (Fig. 6.3).

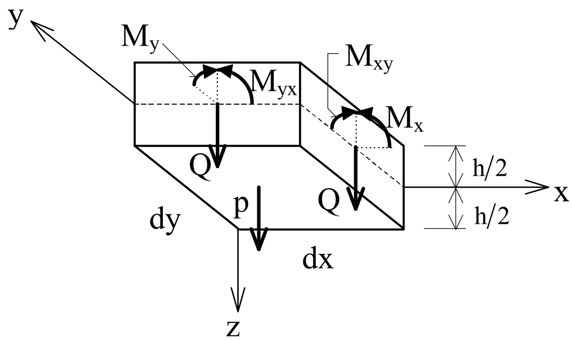


Fig. 6.3. Differential element of the plate strip placed on the half-space with a volume of $dV = dx\ dy\ h$ under the actions of contact stresses p and the section quantities M_x, M_y, M_{xy}, Q_x, Q_y.

$$M_{xy}\left(\xi,t\right)=0$$

$$Q_x\left(\xi,t\right)=-U\left(t\right)*\frac{1}{\left(\frac{\ell}{2}\right)^3}\frac{\partial^3 w\left(\xi,t\right)}{\partial\xi^3} \tag{6.6}$$

$$Q_y\left(\xi,t\right)=0.$$

$$M_x(\xi,t) = -U(t) * \left[\frac{1}{\left(\frac{\ell}{2}\right)^2} \frac{\partial^2 w(\xi,t)}{\partial \xi^2} + (1+\mu_d)\alpha_T \chi_T H(t) \right]$$

$$M_y(\xi,t) = -U(t) * \left[\mu_d \frac{1}{\left(\frac{\ell}{2}\right)^2} \frac{\partial^2 w(\xi,t)}{\partial \xi^2} + (1+\mu_d)\alpha_T \chi_T H(t) \right]$$

For the deflection of the plate strip the following equation is valid

$$\frac{\partial^2}{\partial \xi^2} \left[U(t) * \frac{1}{\left(\frac{\ell}{2}\right)^2} \frac{\partial^2 w(\xi,t)}{\partial \xi^2} \right] = \left(\frac{\ell}{2}\right)^2 p(\xi,t) \tag{6.7}$$

Where

$$U = \frac{h^3}{12(1-\mu_d^2)} R_{d1}^{[1]}(t) \tag{6.8}$$

represents the time function of the flexural stiffness of the strip which is obtained from the flexural stiffness for the elastic problem

$$U = \frac{E_d h^3}{12(1-\mu_d^2)} \tag{6.9}$$

The vertical displacements of the border of the viscoelastic half-space are determined by assuming its planar deformations due to stresses *p(ξ,t)* from the following integral equation

$$v_z(\xi,t) = \frac{(1+\mu_z)\frac{\ell}{2}}{\pi} D_{z1}^{[1]}(t) * \int_{-1}^{+1} p(\bar{\xi},t) \left[2(1-\mu_z)\ln\left(\xi - \bar{\xi} \right(+ \delta(t) \right] d\bar{\xi} + \mathrm{C}(t) \tag{6.10}$$

the elastic adjoint equivalent of which is the well-known Flamant′s equation

$$v_z(\xi) = -\frac{(1+\mu_z)\frac{\ell}{2}}{\pi E_z} \cdot \tag{6.11}$$
$$\cdot \int_{-1}^{+1} p(\bar{\xi}) \left[2(1-\mu_z) \ln \left(\xi - \bar{\xi} \right(+1 \right] d\bar{\xi} + C$$

In Equation (6.10) C(t) represents an arbitrary function of time and in Equation (6.11) an arbitrary constant. Abbreviations in the above equations is given:

$$\chi_T = \frac{12}{h^3} \int_{-h/2}^{h/2} zT(z)\, dz \tag{6.12}$$

Approach to the solution of the integro-differential system of basic equations (6.1), (6.7), (6.10), and (6.2) is used indirectly. As the temperature field is symmetric with regard to the plane $\xi = 0$, also the distribution of the contact stresses $p(\xi,t)$ in foundation joint is symmetric. Therefore, let us chose the functions of the contact stresses in the form as follows

α_T	–	the coefficient of thermal dilatation,
E_d, E_z	–	Young moduli of the plate and of the subgrade materials,
R_{dl}, R_{zl}	–	the transient relaxation functions,
D_{dl}, D_{zl}	–	the transient creep functions of the plate and of the subgrade matrials at the uniaxial stress state,
*	–	symbol that denotes the convolution product,
[1]	–	symbol that denotes the derivation according to the theory of generalized functions (distributive derivation) defined over the time half-axis $t \geq 0$,
$\delta(t)$	–	Dirac′s delta function (distribution),
χ_T	–	the curvature of the middle plane of the strip due to temperature change only.

$$p(\xi,t) = \frac{A(t)}{\sqrt{1-\xi^2}} + \sum_{n=0}^{\infty} a_{2n}(t) P_{2n}(\xi) \quad (n = 0, 1, 2, \dots) \tag{6.13}$$

where $A(t)$ and $a_{2n}(t)$ are the time functions, unknown for the present, and $P_{2n}(\xi)$ are Legendre polynomials of the first order. The n-th Legendre polynomial of the first order is defined as follows

$$P_n(\xi) = \sum_{r=0}^{\left[\frac{n}{2}\right]} \frac{(-1)^r (2n-2r)!}{2^n r!(n-r)!(n-2r)!} \xi^{n-2r} \tag{6.14}$$

$$(n = 0, 1, 2, \dots) \qquad -1 \le \xi \le +1$$

where [n/2] means the integer of the number n/2.

The first equilibrium condition (6.1) leads to the following Equation

$$\pi A(t) + 2a_0(t) = 0 \tag{6.15}$$

The second condition (6.1) is fulfilled identically. As it can be seen from Equation (6.15), the resultant of the contact stresses equals zero and the stresses $p(\xi,t)$ represent a self-loading system. The following relationships between the time functions $A(t)$ and $a_{2n}(t)$ are derivated from the deformation condition (6.2). The deflections of the plate strip $w(\xi,t)$ are composed from two parts: from the deflections $w_p(\xi,t)$ due to contact stresses $p(\xi,t)$ and from the deflections $w_T(\xi,t)$ due to temperature $T(z)H(t)$, i.e.

$$w(\xi,t) = w_p(\xi,t) + w_T(\xi,t) \tag{6.16}$$

It is dealt with the deflections due to contact stresses (6.13). For the plate strip with absolutely stiff borders is given then in the following form

$$\begin{aligned} w_p(\xi,t) = \left(\frac{\ell}{2}\right)^4 U^{*-1}(t) * \Big\{ & A(t)\left\{w_A(\xi)\left[H(\xi)-H(\xi-\beta)\right]+\right. \\ & \left.+\left[w_A(\beta)+(\xi-\beta)w_A'(\beta)\right]\left[H(\xi-\beta)-H(\xi-1)\right]\right\}+ \\ & +\sum_{n=0}^{\infty} a_{2n}(t)\left\{w_{2n}(\xi)\left[H(\xi)-H(\xi-\beta)\right]+\right. \\ & \left.+\left[w_{2n}(\beta)+(\xi-\beta)w_{2n}'(\beta)\right]\left[H(\xi-\beta)-H(\xi-1)\right]\right\} \Big\}, \end{aligned} \tag{6.17}$$

where

$$U^{*-1}(t) = \frac{12\left(1-\mu_d^2\right)}{h^3} D_{d1}^{[1]}(t) \tag{6.18}$$

$(\ell/2)^2$ was the inverse element of the flexural stiffness of the strip plate (6.8), and

$$
\begin{aligned}
w_A(\xi) &= \frac{1}{36}\left[\xi\left(6\xi^2+9\right)\arcsin\xi+\left(11\xi^2+4\right)\sqrt{1-\xi^2}-4\right] \\
w_0(\xi) &= \frac{1}{24}\left(\xi^4+6\xi^2\right) \\
w_2(\xi) &= \frac{1}{240}\left(\xi^6-5\xi^4+15\xi^2\right) \\
w_4(\xi) &= \frac{1}{384}\left(\xi^8-4\xi^6+6\xi^4-4\xi^2\right) \\
w_6(\xi) &= \frac{1}{3840}\left(11\xi^{10}-45\xi^8+70\xi^6-50\xi^4+15\xi^2\right) \\
w_8(\xi) &= \frac{1}{15360}\left(65\xi^{12}-286\xi^{10}+495\xi^8-420\xi^6+175\xi^4-30\xi^2\right) \\
&\vdots
\end{aligned}
\tag{6.19}
$$

were the delection functions of the strip plate due to particular terms of the approximation function (6.13).

The deflections due to temperature have to fulfil the following differential equation

$$
\frac{1}{\left(\frac{\ell}{2}\right)^2}\frac{\partial^2}{\partial\xi^2}\left[U(t)*\frac{1}{\left(\frac{\ell}{2}\right)^2}\frac{\partial^2 w_T(\xi,t)}{\partial\xi^2}\right]=0 \tag{6.20}
$$

as well as the following conditions:

In the middle ($\xi = 0$)

$$
w_T^I(\xi,t)\Big|_{\xi=0}=0 \qquad \frac{\partial w_T^I(\xi,t)}{\partial\xi}\Bigg|_{\xi=0}=0 \qquad Q_{xT}^I(\xi,t)\Big|_{\xi=0}=0 \tag{6.21}
$$

in the discontinuity point of the temperature field ($\xi = \alpha$)

$$
\begin{gathered}
w_T^I(\xi,t)\Big|_{\xi=\alpha}=w_T^{II}(\xi,t)\Big|_{\xi=\alpha} \qquad \frac{\partial w_T^I(\xi,t)}{\partial\xi}\Bigg|_{\xi=\alpha}=\frac{\partial w_T^{II}(\xi,t)}{\partial\xi}\Bigg|_{\xi=\alpha} \\
M_{xT}^I(\xi,t)\Big|_{\xi=\alpha}=M_{xT}^{II}(\xi,t)\Big|_{\xi=\alpha}
\end{gathered}
\tag{6.22}
$$

in the abrupt change of stiffness ($\xi = \beta$)

$$M_{xT}^{II}(\xi,t)\Big|_{\xi=\beta} = M_{xT}^{III}(\xi,t)\Big|_{\xi=\beta} \quad Q_{xT}^{II}(\xi,t)\Big|_{\xi=\beta} = Q_{xT}^{III}(\xi,t)\Big|_{\xi=\beta},$$

and on the border ($\xi = 1$) of the plate strip

$$M_{xT}^{III}(\xi,t)\Big|_{\xi=1} = 0 \qquad Q_{xT}^{III}(\xi,t)\Big|_{\xi=1} = 0 \tag{6.23}$$

The equation (6.20) and the conditions (6.21) ÷ (6.23) is fulfilled by the following function

$$w_T(\xi,t) = \left(\frac{\ell}{2}\right)^2 KH(t)\,\overline{w}_T(\xi) \tag{6.24}$$

where

$$\mathrm{K} = (1+\mu_d)\,\alpha_T \chi_T \tag{6.25}$$

and

$$\begin{aligned}\overline{w}_T(\xi) = &-\frac{1}{2}\xi^2\left[H(\xi)-H(\xi-\alpha)\right] + \frac{1}{2}\left(\alpha^2-2\alpha\xi\right).\\ &.\left[H(\xi-\alpha)-H(\xi-\beta)\right] + \left[\frac{1}{2}\left(\alpha^2-2\alpha\beta\right)-\alpha(\xi-\beta)\right].\\ &.\left[H(\xi-\beta)-H(\xi-1)\right].\end{aligned} \tag{6.26}$$

As it can be seen from Equations (6.24) and (6.25), the deflections due to temperature do not depend on the relaxation properties of the plate material, but on the character of the temperature field only. The influence of the temperature field is expressed by the curvature χ_T. It is given by the formulae (6.12). It depends on the character of the temperature gradient through the plate thickness. The temperature gradient can be linear, but also non linear. In both cases is calculated the curvature χ_T from the formula (6.12). In the case of the linear temperature gradient (Fig. 6.4)

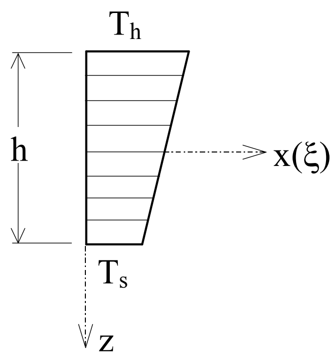


Fig. 6.4. Linear temperature gradient along the plate thickness *h*

$$T\left(z\right)=\frac{\Delta T}{h}z+T_0 \tag{6.27}$$

where $\Delta T = T_s - T_h$ and $T_0 = (T_s + T_h)/2$ is obtained

$$\chi_T=\frac{\Delta T}{h} \tag{6.28}$$

In the case of non linear temperature gradient (Fig. 6.5) is approximated with a sufficint accuracy the temperature change in an arbitrary point of the plate thickness using the following expression

$$T\left(z\right)=az^3+bz^2+cz+d \tag{6.29}$$

where

$$a=\frac{9}{2h^3}\left[T_4-T_1+3\left(T_2-T_3\right)\right] \qquad b=\frac{9}{4h^2}\left[T_1-T_2-T_3+T_4\right]$$

$$c=\frac{1}{8h}\left[T_1-T_4+27\left(T_3-T_2\right)\right] \qquad d=\frac{1}{16}\left[9\left(T_2+T_3\right)-\left(T_1+T_4\right)\right]$$

and T_1, T_2, T_3, T_4 are the discrete temperature values in the planes z = -h/2, -h/6, +h/6, +h/2. The curvature χ_T after substitution of (6.29) into (6.12), and after integration is as follows:

$$\chi_T=\frac{1}{20h}\left[11\left(T_4-T_1\right)+27\left(T_3-T_2\right)\right] \tag{6.30}$$

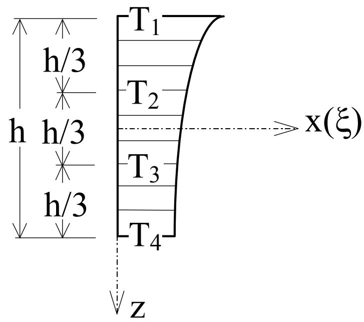


Fig. 6.5. Non linear temperature gradient along the plate thickness *h*

Let us return again to the deformation condition (6.2). The vertical displacements of the half-space border ($z = 0$) are caused only by contact stresses $p(\xi,t)$. They determined them from the integral equation (6.10). If is assumed that the stresses (6.13) have a form (6.13), then the displacements $v_z(\xi,t)$ can be represented in the form as follows

$$v_z(\xi,t) = \frac{\left(1-\mu_z^2\right)\frac{\ell}{2}}{\pi} D_{z1}^{[1]}(t) * \left[\sum_{n=0}^{\infty} a_{2n}(t) v_{2n}(\xi)\right]\left[H(\xi)-H(\xi-1)\right] \quad (n = 0, 1, 2, \dots) \tag{6.31}$$

where

$$v_0(\xi) = (\xi-1)\ln|\xi-1| - (\xi+1)\ln|\xi+1|$$

$$v_2(\xi) = \frac{1}{2}\left[\left(\xi^3-\xi\right)\ln\frac{\|\xi-1\|}{|\xi+1|} + 2\xi^2\right]$$

$$v_4(\xi) = \frac{1}{8}\left[\left(7\xi^5-10\xi^3+3\xi\right)\ln\frac{\|\xi-1\|}{|\xi+1|} + \frac{1}{3}(42\xi^4-46\xi^2)\right]$$

$$v_6(\xi) = \frac{1}{16}\left[\left(33\xi^7-63\xi^5+35\xi^3-5\xi\right)\ln\frac{\|\xi-1\|}{|\xi+1|} + \frac{1}{5}(330\xi^6-520\xi^4+206\xi^2)\right]$$

$$v_8(\xi) = \frac{1}{128}\left[\left(715\xi^9 - 1716\xi^7 + 1386\xi^5 - 420\xi^3 + 35\xi\right)\ln\frac{\|\xi - 1\|}{|\xi + 1|} + \right.$$
$$\left. + \frac{2}{105}(75075\xi^8 - 155155\xi^6 + 100485\xi^4 - 20901\xi^2)\right]$$
$$\vdots$$

are the veritcal displacements of the half-space border due to particular terms of the approximation function (6.13). After expressing the deflections $w(\xi,t)$ and $v_z(\xi,t)$ according to (6.16), (6.27), (6.24), and (6.31), respectively, and after modification, the deformation condition leads to the functional equation for the infinite sequence of the time functions $A(t)$ and $a_{2n}(t)$, as follows:

$$k(t) * A(t)\left\{w_A(\xi)\left(H(\xi) - H(\xi - \beta)\right) + \right.$$
$$\left. + \left(w_A(\beta) + (\xi - \beta)w_A'(\beta)\right)\left(H(\xi - \beta) - H(\xi - 1)\right)\right\} +$$
$$+ \sum_{n=0}^{\infty} a_{2n}(t) * \left\{ \delta(t) v_{2n}(\xi)\left(H(\xi) - H(\xi - 1)\right) + k(t). \right. \tag{6.32}$$
$$.\left[\ w_{2n}(\xi)\left(H(\xi) - H(\xi - \beta)\right) + \left(w_{2n}(\beta) + (\xi - \beta)w_{2n}'(\beta)\right)\right.$$
$$\left. \left. .\ \left(H(\xi - \beta) - H(\xi - 1)\right)\right] \right\} = \frac{1}{\left(\frac{\ell}{2}\right)^2} k(t) * U(t) * KH(t)\overline{w}_T(\xi)$$

This equation is to be fulfilled identically for all points of the interval $-1 \leq \xi \leq +1$ (n = 0, 1, 2, …) and in the each instant $t \in <0, \infty)$. The time function of the relative flexural stiffness of the plate strip is defined by the convolution relationship

$$k(t) = 2\frac{3\left(\frac{\ell}{2}\right)^3 (1 - \mu_d^2)}{h^3 (1 - \mu_z^2)} D_{d1}^{[1]}(t) * R_{z1}^{[1]}(t) \tag{6.33}$$

The equations (6.15) and (6.32) are solved using collocation method in the space of Laplace′s transforms. Applying collocation method are not fulfiled the deformation condition for each ξ of the interval < -1, +1 > of the foundation joint, but in a finite number of the discrete points ξ = 0,2; 0,4; 0,6; 0,8, and 1,0 only. (Equation (6.32) is satisfied for negative arguments ξ because of symmetry. In the middle of the plate strip the equation is satisfied in advance because of the first condition (6.21)). In such a way, are obtained together with the equilibrium equation (6.15) a finite system of six functional equations for six unknown time functions $A(t)$ and $a_{2n}(t)$ ($n = 0, 1, 2, 3, 4$). The time unknown t from in such a way obtained equation system are eliminated by means of the following Laplace integral transformation

$$\mathrm{L}\left\{f(\xi,t)\right\} = \tilde{f}(\xi,\lambda) = \int_0^{\infty} f(\xi,t)e^{-\lambda t}dt) \tag{6.34}$$

where λ is the parameter of the Laplace transformation. Consequently, is started from the following equations

$$\pi\,\tilde{A}(\lambda) + 2\,\tilde{a}_0(\lambda) = 0 \tag{6.35}$$

$$\begin{aligned}&\tilde{k}(\lambda)\tilde{A}(\lambda)\left[w_A(\xi)\left(H(\xi)-H(\xi-\beta)\right)+\left(w_A(\beta)+(\xi-\beta)w_A'(\beta)\right).\right.\\&\left..\left(H(\xi-\beta)-H(\xi-1)\right)\right]+\sum_{n=0}^{\infty}\tilde{a}_{2n}(\lambda)\left\{\;v_{2n}(\xi)\left(H(\xi)-H(\xi-1)\right)\right.\\&+\tilde{k}(\lambda)\left[w_{2n}(\xi)\left(H(\xi)-H(\xi-\beta)\right)+\left(w_{2n}(\beta)+(\xi-\beta)w_{2n}'(\beta)\right).\right.\\&\left.\left..\left(H(\xi-\beta)-H(\xi-1)\right)\right]\;\right\}=\frac{1}{\left(\frac{\ell}{2}\right)^2}\tilde{k}(\lambda)\tilde{U}\left(\lambda\right)K\overline{w}_T(\xi)\frac{1}{\lambda}\end{aligned} \tag{6.36}$$

The Laplace′s transforms of the holomorphic functions of the relative flexural stiffness of the plate strip and its flexural stiffness (6.8) with regard to the transformant

$$\lambda\tilde{R}(\lambda) = \frac{1}{\lambda D(\lambda)}$$

resulting from (Kovařík, 1987) results in the following relations

$$\begin{aligned}\tilde{k}\;(\lambda) &= 2\frac{3\left(\frac{\ell}{2}\right)^3(1-\mu_d^2)\tilde{D}_{d1}(\lambda)}{h^3(1-\mu_z^2)\tilde{D}_{z1}(\lambda)}\\ \tilde{U}(\lambda) &= \frac{h^3}{12(1-\mu_d^2)}\frac{1}{\lambda\tilde{D}_{d1}(\lambda)}\end{aligned} \tag{6.37}$$

The originals $A(t)$ and $a_{2n}(t)$ of the holomorphic functions $\tilde{A}(t)$ and $\tilde{a}(t)$ are constructed applying numerical inverse transformation modified by

Erdélyi – Schapery method. The knowledge of the originals of time functions $A(t)$ and $a_{2n}(t)$ permits to determine the stresses in the foundation joint of the plate strip, its deflections and its internal forces for different instants t. For the contact stress the approximation relationship (6.13) is valid. At the determination of the resultant deformations of the plate strip the following two cases should be distinguished:

- when the material of the strip plate is not subjected to creep, then determining deflections are determined from the following relationship

$$w(\xi,t)=\left(\frac{\ell}{2}\right)^4 U^{-1}\left\{ A(t)\left[w_A(\xi)\left(H(\xi)-H(\xi-\beta)\right)+\right.\right. \tag{6.38}$$

$$+\left(w_A(\beta)+(\xi-\beta)w'_A(\beta)\right).\left(H(\xi-\beta)-H(\xi-1)\right)\Big]+$$

$$+\sum_{n=0}^{\infty} a_{2n}(t)\left[w_{2n}(\xi)\left(H(\xi)-H(\xi-\beta)\right)+\left(w_{2n}(\beta)+\right.\right.$$

$$\left.\left.+(\xi-\beta)w'_{2n}(\beta)\right)\left(H(\xi-\beta)-H(\xi-1)\right)\right]+ \tag{6.38}$$

$$\left.+\frac{1}{\left(\frac{\ell}{2}\right)^2} U\mathrm{K}\bar{w}_T(\xi)\right\},$$

where U and K are the relationships (6.9) and (6.25) valid for an ideally elastic plate.

- when the material of the plate strip is subjected to creep, then for the detemination of the time function of deflections are determined using the inverse transformation of its Laplace′s transforms according to the Erdélyi Schapery method

$$\tilde{w}(\xi,\lambda)=\left(\frac{\ell}{2}\right)^{4}\tilde{U}^{*-1}(\lambda)\left\{\tilde{A}(\lambda)\left[w_A(\xi)\left(H(\xi)-H(\xi-\beta)\right)+\right.\right.$$
$$+\left(w_A(\beta)+(\xi-\beta)w_A'(\beta)\right)\Big].\left(H(\xi-\beta)-H(\xi-1)\right)\Big]+$$
$$+\sum_{n=0}^{\infty}\tilde{a}_{2n}(\lambda)\left[w_{2n}(\xi)\left(H(\xi)-H(\xi-\beta)\right)+\left(w_{2n}(\beta)+\right.\right. \tag{6.39}$$
$$\left.\left.+(\xi-\beta)w_{2n}'(\beta)\right)\left(H(\xi-\beta)-H(\xi-1)\right)\right]+$$
$$\left.+\frac{1}{\left(\frac{\ell}{2}\right)^{2}}\tilde{U}(\lambda)\tilde{\mathrm{K}}\frac{1}{\lambda}\bar{w}_T(\xi)\right\}.$$

At the determination of internal forces the following common relationships are valid in both cases:

$$M_X(\xi,t)=-\left(\frac{\ell}{2}\right)^{2}\left[A(t)w_A''(\xi)+\sum_{n=0}^{\infty}a_{2n}(t)w_{2n}''(\xi)\right].$$
$$.\left[H(\xi)-H(\xi-\beta)\right]$$
$$Q_X(\xi,t)=-\left(\frac{\ell}{2}\right)^{3}\left[A(t)w_A'''(\xi)+\sum_{n=0}^{\infty}a_{2n}(t)w_{2n}'''(\xi)\right]. \tag{6.40}$$
$$.\left[H(\xi)-H(\xi-\beta)\right].$$

It is demonstrated the general procedure of the solution on the thermoviscoelastic analysis of the plate strip in metallurgical operation which is placed in a stabilised temperature field of ambient environment.

2.3. Examples and analysis of the obtained results

Let us consider the plate strip having a constant stiffness ($\beta = 1$) subjected to a high temperature $T(z)H(t)$ on the whole range < -ℓ/2, +ℓ/2 > of its surface (Fig. 6.6).

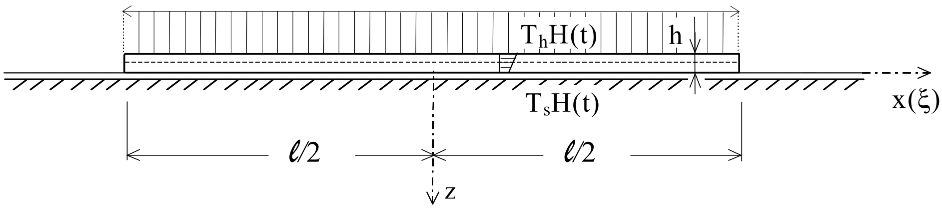


Fig. 6.6. A road plate strip in a stabilized temperature field along the whole strip width ℓ

Stress and deformation states of such a strip are described by Equations (6.38), (6.39), and (6.40) where are determined the time functions $A(t)$ and $a_{2n}(t)$ by inverse Laplace′s transformation of Equations (6.35) and (6.36). In Equation (6.36) the following function corresponds on the basis of (6.26) for $\alpha = 1$ and $\beta = 1$ to the temperature field given in Fig. 6.6:

$$\overline{w}_T(\xi) = \frac{1}{2}\xi^2 \left[H(\xi) - H(\xi - 1) \right] \tag{6.41}$$

and the curvature K is given by the relationship (6.25). As the holomorphic functions $\tilde{k}(\lambda)$ and the functions of the flexural stiffness of the strip $\tilde{U}(\lambda)$, as it can be seen from (6.37), depend on the Laplace′s transforms of the transient creep functions of the materials of the plate strip and of the subgrade, are introduced them for the plate material (concrete) according to J. E. Prokopovič – V. A. Zedgenidze (Prokopovič and Zedgenidze, 1980)

$$D_{d1}(t) = \frac{1}{E_d} + C_0\left(1 - B_1 e^{-\gamma_1 t} - B_2 e^{-\gamma_2 t}\right) \tag{6.42}$$

and for the material of subgrade according to (Meščjan, 1967)

$$D_{z1}(t) = \frac{1}{E_z}\left[1 - \frac{\delta_1}{\delta_2}\left(1 - e^{-\delta_2 t}\right)\right] \tag{6.43}$$

where

E_d, E_z (Mpa), C_0 (MPa^{-1}), B_1, B_2, $\gamma_1\left(1/\text{day}\right)$, $\gamma_2\left(1/\text{day}\right)$, $\delta_1\left(1/\text{day}\right)$, $\delta_2\left(1/\text{day}\right)$, are free parameters among which B_1 and B_2 vary within the limits $0 \leqq B_1 \leqq 1$, $0 \leqq B_2 \leqq 1$ and their sum equals $B_1 + B_2 = 1$. Laplace′s transforms of the functions (6.42) and (6.43) are the following holomorphic function

$$\tilde{D}_{d1}(\lambda) = \frac{1}{E_d}\,\frac{1}{\lambda}\left[1 + E_d C_0\left(1 - B_1 - B_2 + \frac{B_1\gamma_1}{\gamma_1+\lambda} + \frac{B_2\gamma_2}{\gamma_2+\lambda}\right)\right]$$

$$\tilde{D}_{z1}(\lambda) = \frac{1}{E_z}\,\frac{1}{\lambda}\left(1 + \frac{\delta_1}{\delta_2+\lambda}\right). \qquad (6.44)$$

On the basis (6.44) the transformants $\tilde{k}(\lambda)$ and $\tilde{U}(\lambda)$ can by represented in the following final form

$$\tilde{k}(\lambda) = k\,\frac{1 + E_d C_0\left(1 - B_1 - B_2 + \dfrac{B_1\gamma_1}{\gamma_1+\lambda} + \dfrac{B_2\gamma_2}{\gamma_2+\lambda}\right)}{1 + \dfrac{\delta_1}{\delta_2+\lambda}} \qquad (6.45)$$

$$\tilde{U}(\lambda) = U\,\frac{1}{\left[1 + E_d C_0\left(1 - B_1 - B_2 + \dfrac{B_1\gamma_1}{\gamma_1+\lambda} + \dfrac{B_2\gamma_2}{\gamma_2+\lambda}\right)\right]}, \qquad (6.45)$$

where

k – the coefficient of the relative flexural stiffness,

U – the flexural stiffness of the plate strip for the elastic problem given by the relation (6.9),

$$k = 2\,\frac{3\pi\left(1\text{-}\mu_{\mathrm{d}}^2\right)E_{\mathrm{z}}\left(\dfrac{l}{2}\right)^3}{\left(1-\mu_{\mathrm{z}}^2\right)E_{\mathrm{d}}h^3}$$

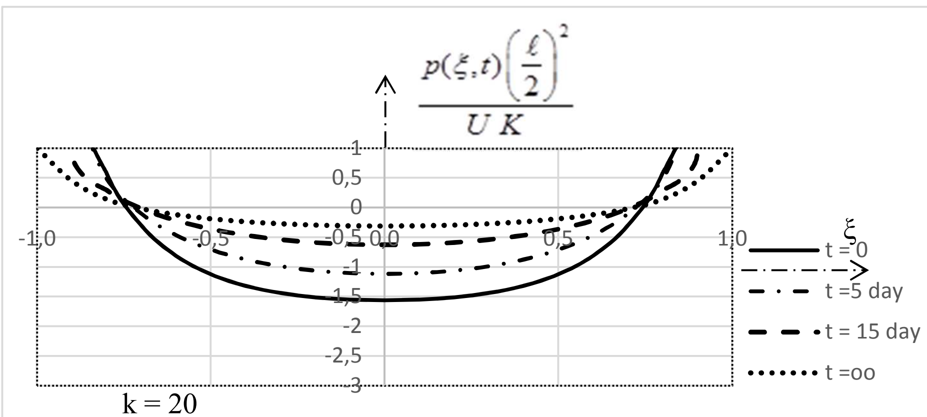
When is satisfied Equation (6.36) only in points ξ = 0,2; 0,4; 0,6; 0,8, and 1,0 at the relative flexural stiffnesss $k = 20$ and at the free parameters $E_d = 2{,}9.10^4$ MPa; $E_z = 4{,}374.10^2$ MPa; $C_0 = 0{,}28.10^{-4}$ MPa^{-1}; $B_1 = 0{,}43$; $B_2 = 0{,}57$; $\gamma_1 = 0{,}0018.1/\mathrm{day}$; $\gamma_2 = 0{,}01.1/\mathrm{day}$; $\delta_1 = 0{,}15.1/\mathrm{day}$, $\delta_2 = 0{,}02.1/\mathrm{day}$, the following inverted values of the roots correspond at instants t = 0, 5, 15, and ∞ days in Table 6.1.

Table 6.1. Inverted values of the roots

The instant	$t = 0$	$t = 5$ days	t = 15 days	$t = \infty$
A	1,992058 L	1,295969 L	0,722833 L	0,606914 L
a_0	-3,129118 L	-2,035704 L	-1,135424 L	-0,953339 L
a_2	1,240038 L	1,006344 L	0,644525 L	0,506457 L
a_4	0,595200 L	0,285634 L	0,115315 L	0,114697 L
a_6	0,100727 L	0,054153 L	0,026399 L	0,024023 L
a_8	0,014467 L	0,008272 L	0,004147 L	0,003678 L

where

$$\mathrm{L}=\frac{UK}{\left(\frac{l}{2}\right)^2}$$



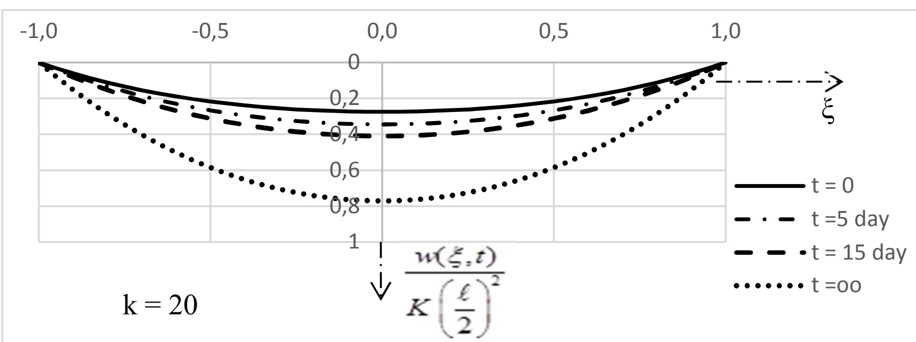
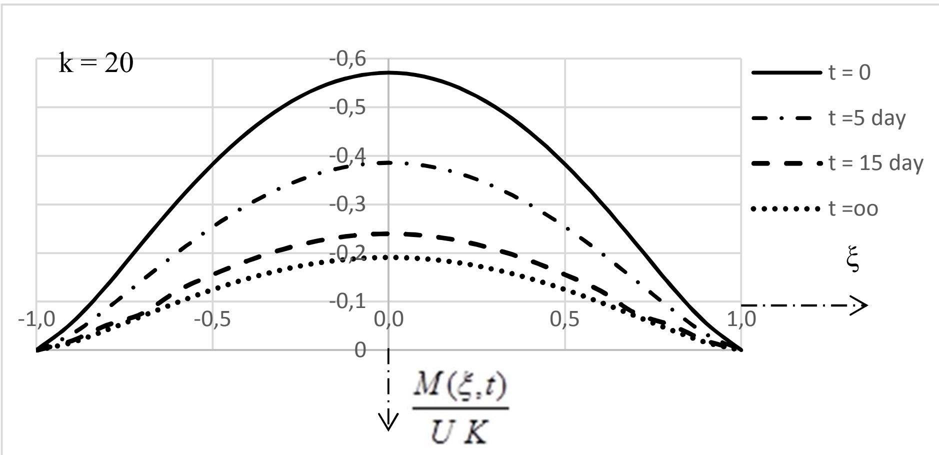


Fig. 6.7. Graphs of contact stresses p(ξ, t) and of deflections w(ξ, t) corresponding to the different instances t due to continuous temperature field according to Fig. 6.6, when the simultaneous creep of plate and of half-space occurs.

The graphs of the corresponding contact stresses, deflections, bending moments, and shear forces are given in Fig. 6.7 and Fig. 6.8. As it can be seen on Fig. 6.7 and Fig. 6.8, while the deflections increase with time, the stresses in the foundation joint, the bending moments and the shear forces decrease rapidly.



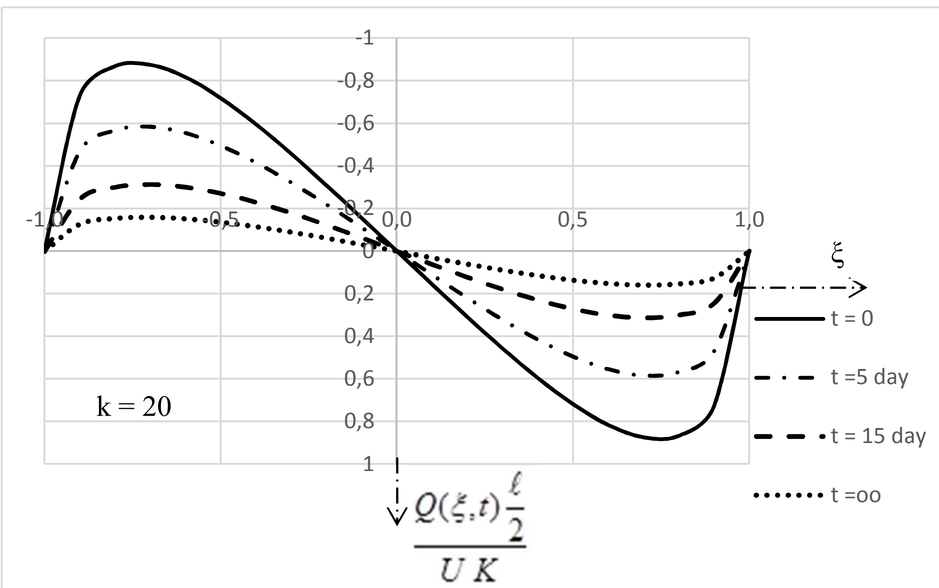


Fig. 6.8. Graphs of bending moments *M(ξ, t)* and of shear forces *Q(ξ, t)* in different instances *t* due to temperature field according to Fig. 6.6, when creep of material both of the plate and that of the subgrade occur simultaneously.

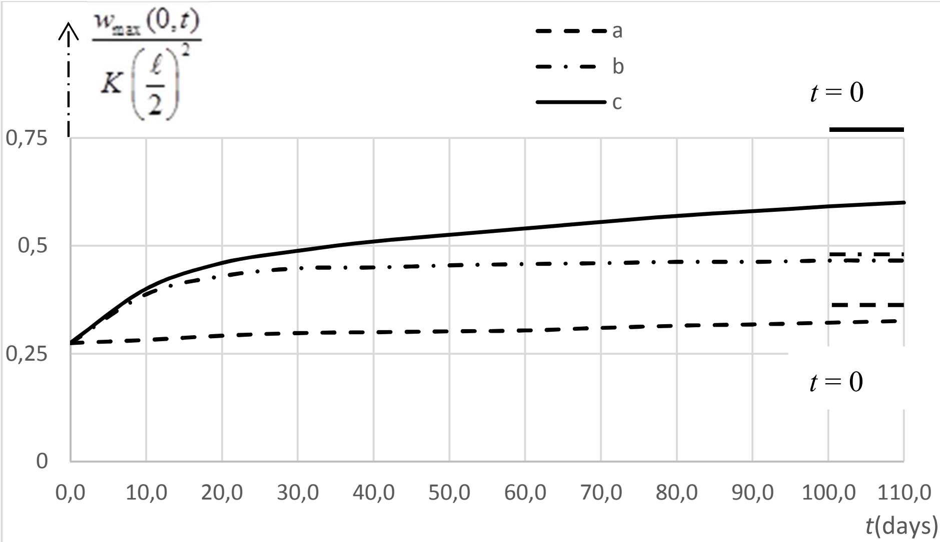


Fig. 6.9. Time dependent change of deflections in the middle of the plate strip, when creep occurs only in the material of plate (curve *a*), when creep occurs only in the material of subgrade (curve *b*), and when creep occurs in the material of plate, as well in that of subgrade simultaneously (curve *c*).

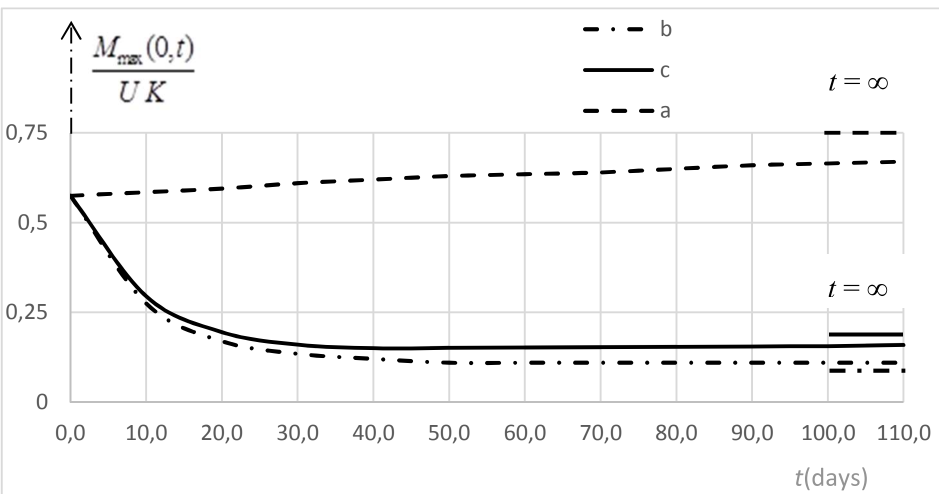


Fig. 6.10. Time dependent change of bending moments in the middle of the plate strip, when creep occurs only in the material of plate (curve *a*), when creep occurs only in the material of subgrade (curve *b*), and when creep occurs in the material of plate, as well as in that of subgrade simultaneously (curve *c*).

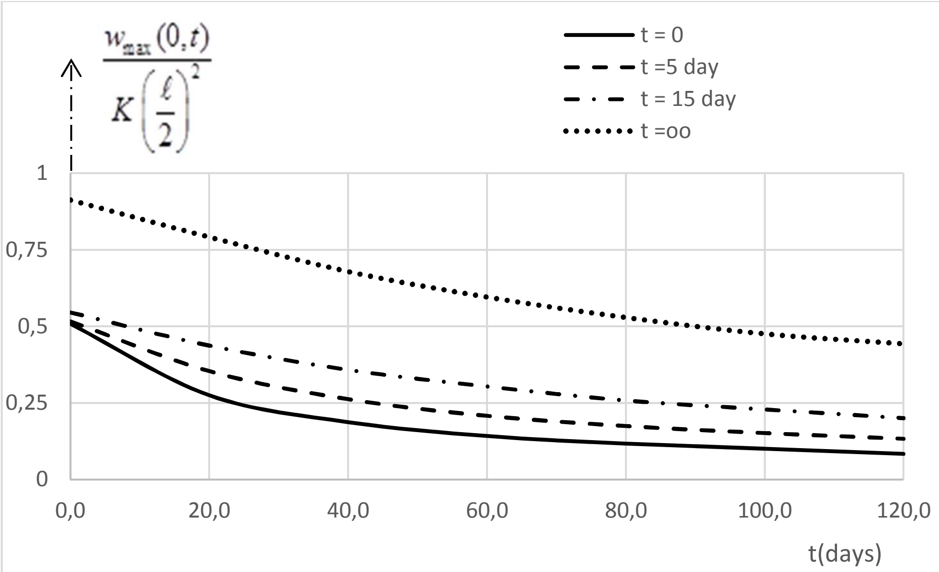


Fig. 6.11. Change of maximum deflections of the plate strip in time *t* at simultaneous creep of plate and of subgrade, when flexibility of the strip is gradually obtaining the following values: *k* = 0, 20, 40, …, 120.

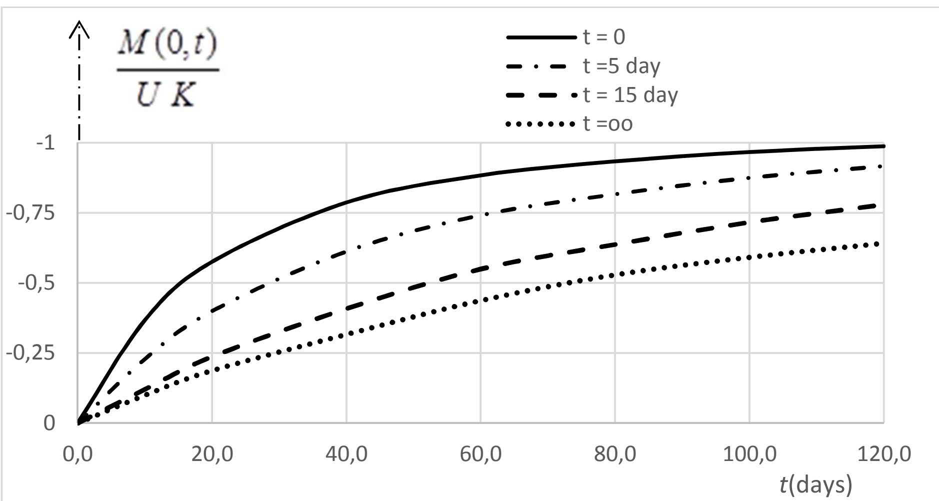


Fig. 6.12. Change of maximum bending moments of the plate strip in time *t* at simultaneous creep of plate and of subgrade, when flexibility of the strip is gradually obtaining the following values: *k* = 0, 20, 40, …, 120.

The maximum deflection in the middle of the plate ($\xi = 0$) at instant $t = \infty$ increased 2,8-times, while the bending moment decreased 2,98-times at the same instant in comparison with the state of its instantaneous warming ($t = 0$). As to shear forces, this decrease represent in the cross section $\xi = 0{,}7$ a multiplier of 3,03. Further, in Fig. 6.9 and Fig. 6.10 the change of the maximum moments and that of bending moments in the middle of the plate strip ($\xi = 0$) on the whole time half-axis $t \in <0, \infty)$ can be monitored at a constant relative flexural stiffness of 20, for three possible cases of creep of the plate and subgrade materials. In Figures is denoted by the letter ***a*** the solution, when only the material of plate was subjected to creep and then is substituted $\delta_1 = 0$ and $\delta_2 = 0$ in the relationships (6.45), further by the letter ***b*** the solution when material of the subgrade showed relaxation abilities and the plate material was elastic ($B_1 = 0$, $B_2 = 0$, $\gamma_1 = 0$, and $\gamma_2 = 0$), and finally by the letter ***c*** the solution when both materials of the plate and of the subgrade was subjected to the creep at the same time with the input data as they was given above. As it can be seen in Fig. 6.9 and Fig. 6.10, while the deflections in all three cases increase gradually, the variation of the bending moments is directly surprising. The change of the maximum deflections and of the bending moments in the middle of the plate is given in Fig. 6.11 and Fig. 6.12 in dependence on the relative flexural stiffness at the simultaneous creep of the materials of the plate and of the subgrade. The area of the change of the mentioned calculation magnitudes is limited by the lines $t = 0$ and $t = \infty$. That area, as it can be seen, is very broad owing to the relaxation properties of plate and subgrade and leads to the contradictory results. While the deflection at time $t = \infty$, at the relative flexural stiffness $k = 120$ increased 5,08-times, the bending moment decreased 1,54-times at the same relative flexural stiffness in comparison with the state of instantaneous warming.

2.4 Conclusions

Stress and deformation states of a plate strip with variable stiffness and Boussinesq half-space subjected to actions of discontinuous temperature field under assumption of plane deformation are investigated, the hereditary creep of the plate and of the subgrade materials according to different constitutive equations is applied. The formulation and the solution of the basic integro-differential equation system are performed in Laplace′s transforms using an undirect way. The time dependent solution is constructed by means of modified Erdélyi - Schapery algorithm. The numerical method with a detailed analysis of the obtained results is demonstrated on a numerical example in the chapter 6.3.

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