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14. Fundamental formulae for the calculation of shear flexible rod structures and some applications

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Abstract: Presented approach concerning analysis of rod structures with shear effects, interacting or not with elastic foundation constitutes consistent and precise solution of the problem. It is characterized by an universal approach in problem description, allowing for analysis of arbitrary particular cases.

Keywords: structural analysis, static, Timoshenko beam, elastic foundation

14.1. Introduction

At present, due to high requirements concerning the results of numerical analysis of structures, the necessity of taking into account effects of shear on calculated values of displacements and stresses is of growing importance – also for rod structures (Filipkowski and Ruchwa, 1991, Pałkowski, 2009).

For typical problems of structural analysis of rod structures subjected to shear effects the Timoshenko formulation is widely applied, possible to implement into Finite Element Method - FEM (Hughes, 2000, Reddy 2006). Common additional problems concern shear locking and elimination of locking (Belytschko, Liu and Moran, 2000) and also introduction of shear effects for various transversal sections of rods (Cowper, 1966, Filipkowski, 1989). The examples of such analyses of frame structures are available in literature (Filipkowski and Ruchwa, 1991), as well as examples for elements taking into account shear effects for frame-strut bars in mast core (Ruchwa and Matuszkiewicz, 2010).

Although the description of shear effects for basic rod elements is evident (but still used rarely), the entire theoretical description for the rod interacting with foundation is still an interesting problem. Many authors proposed solutions obtained using various computational approaches: Finite Differences Method (Sadecka, 2010), Method of Discrete Singular Convolution - DSC (Akgöz *et al.*, 2016) and Finite Element Method (Frydrýšek, Jančo and Gondek, 2013).

Very valuable are formulations assuming the interaction of the bar with two-parameter elastic foundation without shear effects (Teodoru and Muşat, 2008, Dinev, 2012) and with shear ffects (Filipkowski, 1989, 1992, Sienkiewicz and Ruchwa, 1992, Shirima and Giger, 1992).

In this paper the relatively simple and consistent approach taking into account the analytic solution for bars with shear effects, with- or without interaction with two-parameter elastic foundation. This solution was obtained by Filipkowski (Filipkowski, 1989) and developed in following years (Filipkowski and Ruchwa, 1991, Filipkowski and Shirima, 1991, Sienkiewicz and Ruchwa, 1992, Filipkowski, 1992, Shirima and Giger, 1992).

14.2. Theoretical background

The considered structure is a beam of monosymmetric cross-section, with shear effects, located on two-parameter elastic foundation. Linear-elastic material model is assumed, as well as small displacement analysis (Fig. 14.1). Details concerning foundation will not be discussed, because necessary information can be found in many studies (Dembicki *et al.*, 1988, El-Garhy and Osman, 2002).

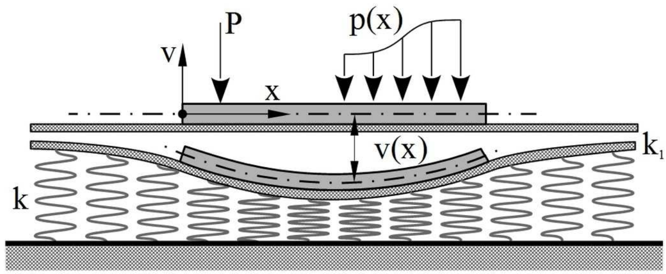


Fig. 14.1. Timoshenko beam on two-parameter elastic foundation

Entire potential energy of the beam is defined by a functional:

$$\begin{aligned}\Pi_c(v,\varphi) &= \tfrac{1}{2}\int_0^L \left(EJ(\varphi')^2 + \kappa GA(v'-\varphi)^2 - p(x)v\right)dx + \\ &+ \tfrac{1}{2}\int_0^L \left(kv^2 + k_1\varphi^2\right)dx - \left(Tv + M\varphi\right)\Big|_0^L\end{aligned} \tag{14.1}$$

where the following parameters are used: vertical displacement of the beam (v), flexural angle of rotation (φ), longitudinal modulus of elasticity (E), transversal modulus (G), section area (A), moment of inertia (J), shear coefficient (κ) of cross-section of the beam, as well as parameters of elastic foundation (k, k_1), length of the beam (L), distributed load (p(x)) and loads on beam ends (T, M).

Due to principle of potential energy minimum the state of static equilibrium is defined by the equation:

$$\delta \Pi_c(v,\varphi) = 0 \tag{14.2}$$

which gives the following differential equation:

$$EJ\,f''''(x) - (k_1 + \eta k\,EJ)\,f''(x) + k(1+\eta k_1)\,f(x) = p(x) \tag{14.3}$$

where:

$$\eta = 1/(\kappa GA) \tag{14.4}$$

The geometrical and statical quantities are defined by relations:

$$v(x) = (1+\eta k_1)f(x) - \eta EJf''(x) \tag{14.5}$$

$$\varphi(x) = f'(x) \tag{14.6}$$

$$\psi(x) = v' - \varphi = \eta k_1 f'(x) - \eta EJf'''(x) \tag{14.7}$$

$$T(x) = -EJf'''(x) + k_1 f'(x) \tag{14.8}$$

$$M(x) = EJf''(x) \tag{14.9}$$

where:

$$f(x) = D_1\Phi_1 + D_2\Phi_2 + D_3\Phi_3 + D_4\Phi_4 \tag{14.10}$$

is the solution of homogeneous differential equation, where the constants are defined by relations 14.5 ÷ 14.9, and the final result may be written in a following form:

$$\begin{Bmatrix} v(x) \\ \varphi(x) \\ T(x) \\ M(x) \end{Bmatrix} = \begin{bmatrix} B_{vv} & B_{v\varphi} & B_{vT} & B_{vM} \\ B_{\varphi v} & B_{\varphi\varphi} & B_{\varphi T} & B_{\varphi M} \\ B_{Tv} & B_{T\varphi} & B_{TT} & B_{TM} \\ B_{Mv} & B_{M\varphi} & B_{MT} & B_{MM} \end{bmatrix} \begin{Bmatrix} v_i \\ \varphi_i \\ T_i \\ M_i \end{Bmatrix} \tag{14.11}$$

or

$$\{V(x)\} = \left[\{B_{\bullet v}\} \quad \{B_{\bullet\varphi}\} \quad \{B_{\bullet Q}\} \quad \{B_{\bullet M}\}\right]\{V_i\} \tag{14.12}$$

and in a brief form

$$\{V(x)\} = [B(x)]\{V_i\} \tag{14.13}$$

where:

$\{V_i\}$ and $\{V(x)\}$ are the state vectors (initial and resulting),

$[B(x)]$ is the transfer matrix.

If the beam is loaded on its length (Fig. 14.2) it is necessary to consider the additional vector of load $\{C(x)\}$

$$\{V(x)\} = [B(x)]\{V_i\} + \{C(x)\} \tag{14.14}$$

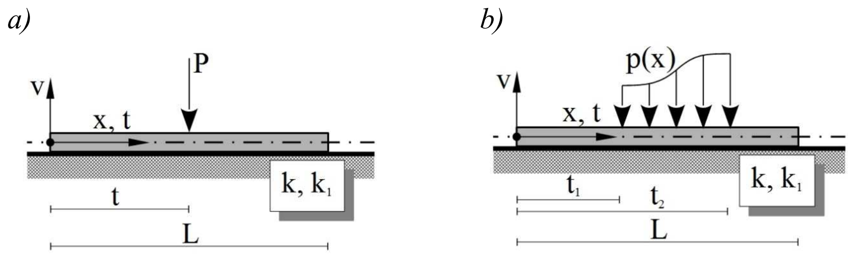


Fig. 14.2. Beam load a) concentrated force, b) distributed load.

Applying Macaulay bracket

$$B\langle x-t\rangle = \begin{cases} 0 &, \quad x<t \\ B(x-t), & \quad x \geq t \end{cases} \tag{14.15}$$

for the concentrated force (Fig. 14.2a), relation 14.14 may be written as follows:

$$\{V(x)\} = [B(x)]\{V_i\} + \{B_{\bullet Q}\langle x-t\rangle\}\, P \tag{14.16}$$

For the distributed load (Fig. 14.2b), his relation has a following form:

$$\{V(x)\} = [B(x)]\{V_i\} + \int_{t_1}^{x}\{B_{\bullet Q}(x-t)\}\, p(t)dt \tag{14.17}$$

Generally, for arbitrary load, the equation 14.14 may be applied to describe the kinematic and static entities inside the beam element, included a complex structure, as defined in matrix description of displacement method or Finite Element Method (Pietrzak, Rakowski and Wrześniowski, 1986, Megson, 2014, Kassimali, 2012). Equation 14.14 may be applied also to describe the relations between state vectors in initial and final node of beam element (in its own local coordinates) (Fig. 14.3) taking into account the internode influence load vector as

$$\{V_k\} = [B(L)]\{V_i\} + \{C\} \tag{14.18}$$

where:

$$\{C\} = \{C_v \quad C_\varphi \quad C_T \quad C_M\}^T \tag{14.19}$$

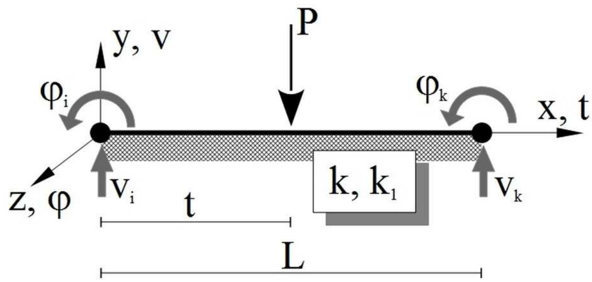


Fig. 14.3. Timoshenko beam element on two-parameter foundation with general internode load.

For concentrated force we obtain:

$$\{C\} = \{B_{\bullet T}(L-t)\}\,P \tag{14.20}$$

and for distributed load:

$$\{C\} = \int_{t_1}^{t_2} \{B_{\bullet T}(L-t)\}\,p(t)dt \tag{14.21}$$

Relation 14.18 can be rewritten:

$$\begin{Bmatrix} \{\delta_k\} \\ \{f_k\} \end{Bmatrix} = \begin{bmatrix} [B_{ii}] & [B_{ii}] \\ [B_{ii}] & [B_{ii}] \end{bmatrix} \begin{Bmatrix} \{\delta_i\} \\ \{f_i\} \end{Bmatrix} + \begin{Bmatrix} \{C_\delta\} \\ \{C_f\} \end{Bmatrix} \tag{14.22}$$

and leads to the following form:

$$\begin{aligned} &\begin{bmatrix} -[B_{ik}]^{-1}[B_{ii}] & [B_{ik}]^{-1} \\ [B_{ki}]-[B_{kk}][B_{ik}]^{-1}[B_{ii}] & [B_{kk}][B_{ik}]^{-1} \end{bmatrix} \begin{Bmatrix} \{\delta_i\} \\ \{\delta_k\} \end{Bmatrix} = \begin{Bmatrix} \{f_i\} \\ \{f_k\} \end{Bmatrix} + \\ &+ \begin{bmatrix} [B_{ik}]^{-1} & [0] \\ [B_{kk}][B_{ik}]^{-1} & -[1] \end{bmatrix} \begin{Bmatrix} \{C_\delta\} \\ \{C_f\} \end{Bmatrix} \end{aligned} \tag{14.23}$$

which is known as equilibrium equation for the element:

$$\left[k_e\right]\left\{\delta_e\right\}=\left\{f_e\right\}+\left\{f_e^0\right\} \tag{14.24}$$

used in matrix displacement method and Finite Element Method, where:

$\left[k_e\right]$ - stiffness matrix,

$\left\{\delta_e\right\}$ - displacement vector,

$\left\{f_e\right\}$ - vector of nodal forces,

$\left\{f_e^0\right\}$ - vector of resultant internodal loads for the element.

Advantegously is to use the element's stiffness matrix in the following form:

$$\left[k_e\right]=\frac{EJ}{L^3}\begin{bmatrix} \gamma & \nu L & -\varepsilon & \delta L \\ \nu L & \alpha L^2 & -\delta L & \beta L^2 \\ -\varepsilon & -\delta L & \gamma & -\nu L \\ \delta L & \beta L^2 & -\nu L & \alpha L^2 \end{bmatrix} \tag{14.25}$$

and the vector of resultant internodal loads as:

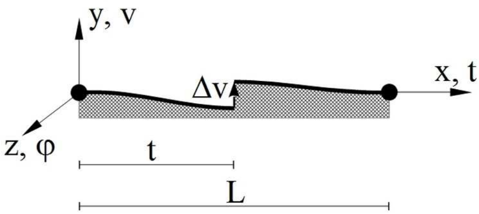
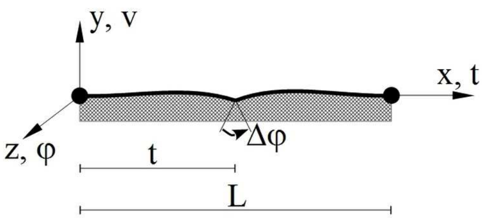
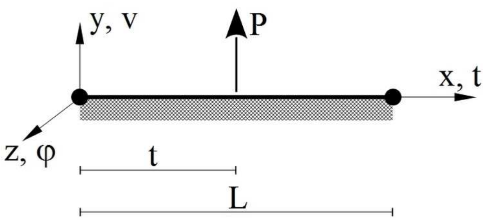
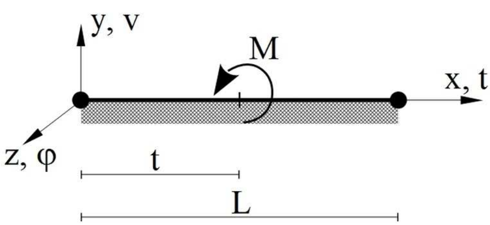
$$\left\{f_e^0\right\}=\begin{Bmatrix} T_i^0 \\ M_i^0 \\ T_k^0 \\ M_k^0 \end{Bmatrix}=\frac{EJ}{L^3}\begin{bmatrix} -\varepsilon & \delta L & 0 & 0 \\ -\delta L & \beta L^2 & 0 & 0 \\ \gamma & -\nu L & -L^3/EJ & 0 \\ -\nu L & \alpha L^2 & 0 & -L^3/EJ \end{bmatrix}\begin{Bmatrix} C_v \\ C_\varphi \\ C_T \\ C_M \end{Bmatrix} \tag{14.26}$$

Relations 14.26 and 14.14 show the possibility to consider the arbitrary internodal loads. Examples of concentrated loads are show in Table 1. For similar distributed loads the equations 14.20 and 14.21 should be used.

The obtained equations 14.14 and 14.24 also 14.25 and 14.26 allow to apply the solution introducing the discretization of the structure known from Finite Element Method, and the implementation of above mentioned equations into realization of these methods.

Of course the given description may be in a simple way completed with the state of loads corresponding to bending in perpendicular plane, longitudinal load and torsion. Due to known solutions concerning these problems they will not be considered in this study (Cook, 2002, Megson, 2014, Akgöz *et al.*, 2016).

Table. 14.1. Loads and corresponding {C}

Load case	{C}
	$\{C\} = \{B_{\bullet v}(L-t)\}\,\Delta v$
	$\{C\} = \{B_{\bullet \varphi T}(L-t)\}\,\Delta\varphi$
	$\{C\} = \{B_{\bullet T}(L-t)\}\,P$
	$\{C\} = \{B_{\bullet M}(L-t)\}\,M$

14.3. Influence of shear for rods without foundation

Presented relations in this case lead to known equations concerning the Timoshenko beam element. Transfer matrix has a following form:

$$\left[B(x)\right] = \begin{bmatrix} 1 & x & \frac{x}{6\,EJ}\left(x^2 - 6\,\mu\,L^2\right) & \frac{-x^2}{2\,EJ} \\ 0 & 1 & \frac{x^2}{2\,EJ} & \frac{-x}{EJ} \\ 0 & 0 & -1 & 0 \\ 0 & 0 & x & -1 \end{bmatrix} \tag{14.27}$$

where

$$\mu = \eta \frac{EJ}{L^2} = \frac{EJ}{\kappa GA\,L^2} \tag{14.28}$$

and the coefficients of element stiffness matrix:

$$\alpha = 4\,\frac{1+3\mu}{1+12\mu} \qquad \beta = 2\,\frac{1-6\mu}{1+12\mu}$$

$$\delta = \nu = 6\,\frac{1}{1+12\mu} \qquad \gamma = \varepsilon = 12\,\frac{1}{1+12\mu} \tag{14.29}$$

For Bernoulli-Euler beam theory the $\mu = 0$ should be assumed.

Describing the influence of shear, the important factor is the shear stiffness, defined in a literature as one of four possible denominations

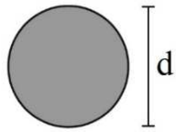
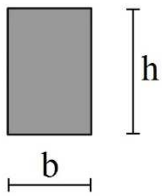
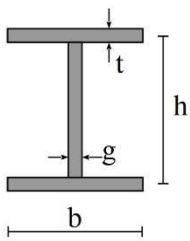
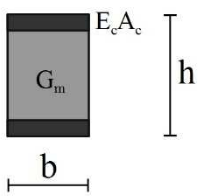
$$\kappa GA = \frac{GA}{k} = GA_z = S_v \tag{14.30}$$

The shear coefficient (κ) can be calculated from the following relation:

$$\kappa = \frac{J^2}{A \int_A \frac{S^2}{b^2}\,dA} \tag{14.31}$$

In Table 14.2. the examples of various shear coefficients are given.

Table. 14.2. Examples of shear coefficients

Section	Shape	Coefficient
Circle (*)		$\kappa = \dfrac{6(1+\nu)}{7+6\nu}$
Rectangle (*)		$\kappa = \dfrac{10(1+\nu)}{12+11\nu}$
Thin-Walled I - Section (*)		$\kappa = 10(1+\nu)(1+3m)^2 / p$, $m = 2bt / hg$, $n = b / h$, $p = 12 + 72m + 150m^2 + 90m^3 +$ $+\nu(11 + 66m + 135m^2 + 90m^3) +$ $+30n^2(m + m^2) +$ $+5\nu n^2(8m + 9m^2)$
Sandwich (**)		$\mu = \dfrac{E_cA_ch}{G_mb}$
Truss (***)		$S_v = EA_k \sin^2\alpha \cos\alpha$

(*) – Cowper G.R. (Cowper, 1966),
(**) – Filipkowski J. (Filipkowski, 1989),
(***) – Pałkowski Sz. (Pałkowski, 2009).

In order to describe the influence of shear, the adequate numerical analyses have been performed for the cantilever beam loaded with concentrated force (Fig. 14.4). Steel I-section has been assumed, similar to I240, with following characteristics: $E = 210$ GPa, $G = 81$ GPa, $A = 46{,}111$ cm^2, $J = 4253{,}3$ cm^4, $\kappa = 0{,}4423$ (calculated from equation 14.31) and variable L (Table 14.3), in relation to assumed L/h, where h = 240 mm.

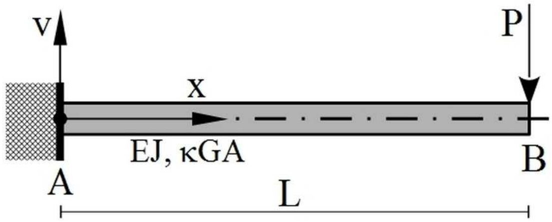


Fig. 14.4. Cantilever beam loaded with concentrated force

Calculations of displacement (v) in B have been performed according the Timoshenko beam (v^T) and Bernoulliego-Eulera (v^{BE}) theories, consistent equations 14.27 ÷ 14.31 using MATLAB system, applied also in FEM analyses (Ferreira, 2009).

As reference results, in order to evaluate the consistence of solution the 3D Finite Element Method analysis has been performed (v^{FEM}). In discrete numerical model the symmetry of the structure was applied. Three-dimensional brick elements have been applied in discrete FEM model, for each mesh built with hundreds thousands elements. Symmetry of the model and load relative to the vertical plane has been assumed. Static linear analyses were performed using ABAQUS (SIMULIA, 2014) computer code. The examples of obtained FEM results are show in Fig. 14.5.

In Table 14.3. the displacements (v^{FEM}, v^T and v^{BE}) and values of relative percentage errors for Timoshenko ($\delta v^T{}_{FEM}$) oraz Bernoulli-Euler ($\delta v^{BE}{}_{FEM}$) models (in relations to FEM solution) as well as solution error of Bernoulli-Euler model in relation to Timoshenko ($\delta v^{BE}{}_T$) model.

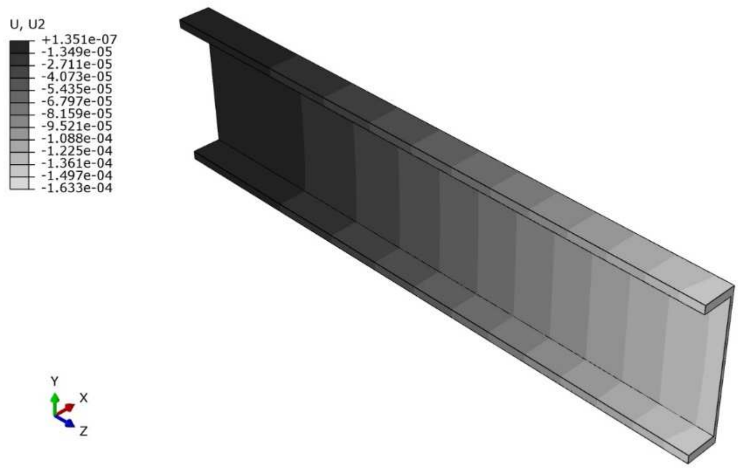


Fig. 14.5. Distribution of vertical displacements obtained by FEM analysis, for L/h=4 (values in meters)

Table. 14.3. Displacements and relative percentage errors

L/h [-]	v^{FEM} [mm]	v^{T} [mm]	v^{BE} [mm]	$\delta v^{T}{}_{FEM}$ [%]	$\delta v^{BE}{}_{FEM}$ [%]	$\delta v^{BE}{}_{T}$ [%]
12	3,6320	3,6356	3,5660	-0,10	1,82	1,92
10	2,1194	2,1217	2,0636	-0,11	2,63	2,74
8	1,1018	1,1031	1,0566	-0,12	4,10	4,22
6	0,4800	0,4806	0,4457	-0,12	7,14	7,26
5	0,2866	0,2870	0,2580	-0,12	10,01	10,12
4 (*)	0,1551	0,1553	0,1321	-0,14	14,84	14,97
3 (*)	0,0730	0,0732	0,0557	-0,24	23,65	23,83

(*) – In author's opinion, beam theory can be used for L/h above 5, results for L/h = 3 and 4 are shown only for demonstrative purposes.

The good consistence between FEM solutions and results for Timoshenko beam can be observed, even for L/h < 5. Results for Bernoulli-Euler model, for shorter cantilevers have growing error values (to 24%).

Assuming the shear coefficient $\kappa = 0{,}4055$, calculated according Cowper G.R. (Cowper, 1966), see Table 14.3, values δv^{T}_{FEM} are growing, and in relation to L/h have the values from -0,28% to -2,41%. If we assume the approximate value of coefficient $\kappa = 0{,}4034$ (calculated as relation of the web area to the total section area) the values of δv^{T}_{FEM} will slightly grow and have the values between -0,29% and -2,55%.

More examples of differencies between Timoshenko and Bernoulli-Euler models are available in publication of Filipkowski and Ruchwa (Filipkowski and Ruchwa, 1991).

14.4. Beams on elastic foundation

Looking for solution in the case of the beam interacting with two-parameter elastic foundation, in order to solve the differentia equation 14.3 it is necessary to solve the homogeneous equation with auxiliary coefficients:

$$b_1 = (k_1 + \eta k EJ)/EJ \qquad b_2 = (1 + \eta k_1) k/EJ \qquad b_3 = 4 b_2 / b_1^2 \qquad (14.32)$$

According to the values of parameter b_3, the equation has two variants (cases) of solution:

- Variant I (a.k.a. *strong foundation*) - $b_3 \geq 1$ (14.33)
- Variant II (a.k.a. *weak foundation*) - $b_3 < 1$ (14.34)

If we assume additional parameters, defined with equations

$$\lambda = \sqrt[4]{(1 + \eta k_1) k/EJ} \qquad \Lambda = \lambda L \qquad (14.35)$$

$$k_2 = \lambda^2 EJ \qquad k_d = k_1 + k_2 \qquad k_r = k_1 - k_2 \qquad (14.36)$$

$$A = 1 + \eta k_d \qquad B = 1 + \eta k_r \qquad C = 1 + \eta k_1 \qquad (14.37)$$

the relations 14.35 ÷ 14.51. will give the exact transfer and stiffness matrices for both variants of solution for Timoshenko beam on two-parameter elastic foundation.

Variant I (strong foundation)

$$a = \sqrt{\frac{1}{2}\left(1 - \frac{b_1}{2\sqrt{b_2}}\right)} \qquad b = \sqrt{\frac{1}{2}\left(1 + \frac{b_1}{2\sqrt{b_2}}\right)} \qquad S = 2ab \tag{14.38}$$

$$\begin{aligned} \phi_1 &= \sin(a\lambda x)\sinh(b\lambda x) & \phi_2 &= \sin(a\lambda x)\cosh(b\lambda x) \\ \phi_3 &= \cos(a\lambda x)\sinh(b\lambda x) & \phi_4 &= \cos(a\lambda x)\cosh(b\lambda x) \end{aligned} \tag{14.39}$$

Table. 14.5. Transfer matrix (for strong foundation)

$\{B_{\bullet v}\}$ (14.40)	$\{B_{\bullet \varphi}\}$ (14.41)
$B_{vv} = \frac{a^2A - b^2B}{SC}\phi_1 + \phi_4$ $B_{\varphi v} = -\frac{b\lambda}{SC}\phi_2 + \frac{a\lambda}{SC}\phi_3$ $B_{Tv} = -\frac{b\lambda k_r}{SC}\phi_2 + \frac{a\lambda k_d}{SC}\phi_3$ $B_{Mv} = -\frac{k_2}{SC}\phi_1$	$B_{v\varphi} = \frac{b}{S\lambda}\phi_2 + \frac{a}{S\lambda}\phi_3$ $B_{\varphi\varphi} = \frac{a^2 k_d + b^2 k_r}{Sk_2}\phi_1 + \phi_4$ $B_{T\varphi} = \frac{a^2 k_d^2 + b^2 k_r^2}{Sk_2}\phi_1$ $B_{M\varphi} = \frac{bk_r}{S\lambda}\phi_2 + \frac{ak_d}{S\lambda}\phi_3$
$\{B_{\bullet T}\}$ (14.42)	**$\{B_{\bullet M}\}$ (14.43)**
$B_{vT} = \frac{bB}{S\lambda k_2}\phi_2 - \frac{aA}{S\lambda k_2}\phi_3$ $B_{\varphi T} = \frac{1}{Sk_2}\phi_1$ $B_{TT} = \frac{a^2 k_d + b^2 k_r}{Sk_2}\phi_1 - \phi_4$ $B_{MT} = \frac{b}{S\lambda}\phi_2 + \frac{a}{S\lambda}\phi_3$	$B_{vM} = -\frac{1}{Sk_2}\phi_1$ $B_{\varphi M} = -\frac{bB\lambda}{SCk_2}\phi_2 - \frac{aA\lambda}{SCk_2}\phi_3$ $B_{TM} = -\frac{b\lambda}{SC}\phi_2 + \frac{a\lambda}{SC}\phi_3$ $B_{MM} = \frac{a^2A - b^2B}{SC}\phi_1 - \phi_4$

Wariant II (weak foundation)

$$a=\sqrt{\frac{1}{2}\left(-1+\frac{b_1}{2\sqrt{b_2}}\right)}\qquad b=\sqrt{\frac{1}{2}\left(1+\frac{b_1}{2\sqrt{b_2}}\right)}\qquad S=2ab \tag{14.44}$$

$$\begin{aligned}\phi_1&=\sinh(a\lambda x)\sinh(b\lambda x) & \phi_2&=\sinh(a\lambda x)\cosh(b\lambda x)\\ \phi_3&=\cosh(a\lambda x)\sinh(b\lambda x) & \phi_4&=\cosh(a\lambda x)\cosh(b\lambda x)\end{aligned} \tag{14.45}$$

Table. 14.6. Transfer matrix (for weak foundation)

$\{B_{\bullet v}\}$ (14.46)	$\{B_{\bullet\varphi}\}$ (14.47)
$B_{vv}=-\frac{a^2A+b^2B}{SC}\phi_1+\phi_4$ $B_{\varphi v}=-\frac{b\lambda}{SC}\phi_2+\frac{a\lambda}{SC}\phi_3$ $B_{Tv}=-\frac{b\lambda k_r}{SC}\phi_2+\frac{a\lambda k_d}{SC}\phi_3$ $B_{Mv}=-\frac{k_2}{SC}\phi_1$	$B_{v\varphi}=\frac{b}{S\lambda}\phi_2+\frac{a}{S\lambda}\phi_3$ $B_{\varphi\varphi}=\frac{-a^2k_d+b^2k_r}{Sk_2}\phi_1+\phi_4$ $B_{T\varphi}=\frac{-a^2k_d^2+b^2k_r^2}{Sk_2}\phi_1$ $B_{M\varphi}=\frac{bk_r}{S\lambda}\phi_2+\frac{ak_d}{S\lambda}\phi_3$
$\{B_{\bullet T}\}$ (14.48)	**$\{B_{\bullet M}\}$ (14.49)**
$B_{vT}=\frac{bB}{S\lambda k_2}\phi_2-\frac{aA}{S\lambda k_2}\phi_3$ $B_{\varphi T}=\frac{1}{Sk_2}\phi_1$ $B_{TT}=\frac{-a^2k_d+b^2k_r}{Sk_2}\phi_1-\phi_4$ $B_{MT}=\frac{b}{S\lambda}\phi_2+\frac{a}{S\lambda}\phi_3$	$B_{vM}=-\frac{1}{Sk_2}\phi_1$ $B_{\varphi M}=-\frac{bB\lambda}{SCk_2}\phi_2-\frac{aA\lambda}{SCk_2}\phi_3$ $B_{TM}=-\frac{b\lambda}{SC}\phi_2+\frac{a\lambda}{SC}\phi_3$ $B_{MM}=-\frac{a^2A+b^2B}{SC}\phi_1-\phi_4$

Table. 14.7. Stiffness matrix for element on elastic foundation

$\left[\mathbf{k}_e\right]$ - for strong foundation	(14.50)
$\alpha=\Lambda\dfrac{aA\sinh(b\Lambda)\cosh(b\Lambda)-bB\sin(a\Lambda)\cos(a\Lambda)}{\Delta}SC$ $\beta=-\Lambda\dfrac{aA\sinh(b\Lambda)\cos(a\Lambda)-bB\sin(a\Lambda)\cosh(b\Lambda)}{\Delta}SC$ $\gamma=\Lambda^3\dfrac{aA\sinh(b\Lambda)\cosh(b\Lambda)+bB\sin(a\Lambda)\cos(a\Lambda)}{\Delta}S$ $\delta=\Lambda^2\dfrac{\sin(a\Lambda)\sinh(b\Lambda)}{\Delta}SC$ $\varepsilon=\Lambda^3\dfrac{aA\sinh(b\Lambda)\cos(a\Lambda)+bB\sin(a\Lambda)\cosh(b\Lambda)}{\Delta}S$ $\nu=\Lambda^2\dfrac{a^2A\sinh^2(b\Lambda)+b^2B\sin^2(a\Lambda)}{\Delta}$ $\Delta=\left(aA\sinh(b\Lambda)\right)^2-\left(bB\sin(a\Lambda)\right)^2$	
$\left[\mathbf{k}_e\right]$ - for weak foundation	(14.51)
$\alpha=\Lambda\dfrac{aA\sinh(b\Lambda)\cosh(b\Lambda)-bB\sinh(a\Lambda)\cosh(a\Lambda)}{\Delta}SC$ $\beta=-\Lambda\dfrac{aA\sinh(b\Lambda)\cosh(a\Lambda)-bB\sinh(a\Lambda)\cosh(b\Lambda)}{\Delta}SC$ $\gamma=\Lambda^3\dfrac{aA\sinh(b\Lambda)\cosh(b\Lambda)+bB\sinh(a\Lambda)\cosh(a\Lambda)}{\Delta}S$ $\delta=\Lambda^2\dfrac{\sinh(a\Lambda)\sinh(b\Lambda)}{\Delta}SC$ $\varepsilon=\Lambda^3\dfrac{aA\sinh(b\Lambda)\cosh(a\Lambda)+bB\sinh(a\Lambda)\cosh(b\Lambda)}{\Delta}S$ $\nu=\Lambda^2\dfrac{a^2A\sinh^2(b\Lambda)+b^2B\sinh^2(a\Lambda)}{\Delta}$ $\Delta=\left(aA\sinh(b\Lambda)\right)^2-\left(bB\sinh(a\Lambda)\right)^2$	

14.5. Examples of Timoshenko beam on elastic foundation

In order to show the possibilities of the considered algorithm, an example of a beam with shear effects and located on two-parameter elastic foundation was analyzed (Fig. 14.6).

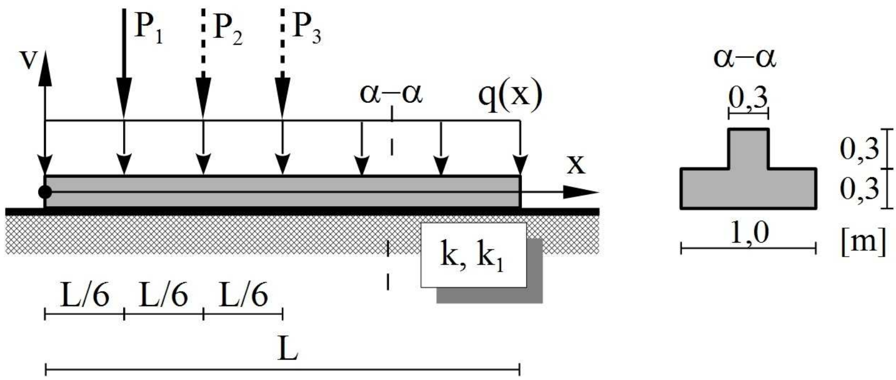


Fig. 14.6. An example of a Timoshenko beam on two-parameter elastic foundation.

Assumed characteristics of the beam:

- E = 33GPa, G = 14 GPa (concrete C30/37),
- A = 3900 cm^2, J = 915580 cm^4, κ = 0,7008 (T-section),
- L = 3 m / 6 m / 12 m (L/h = 5 / 10 / 20);

characteristics of foundation:

- k = 80 MN/m
- k_1 = 20 MN;

applied load:

- q = 10 kN/m (own weight),
- $P_1 = P_2 = P_3$ = 750 kN (concentrated load).

Various configurations of analyzed example have been assumed: 3 variants of length, 3 variants of concentrated force P_i localization, two computational models was also assumed: beam with shear effects on two-parameter elastic foundation (T2) and beam without shear effects on Winkler foundation (BE1). All calculations were performed using MATLAB (MathWorks, 2015) code, applying the matrix version of displacements method and described relations. On Figs 14.7 ÷ 14.10 the obtained kinematic and static results are presented.

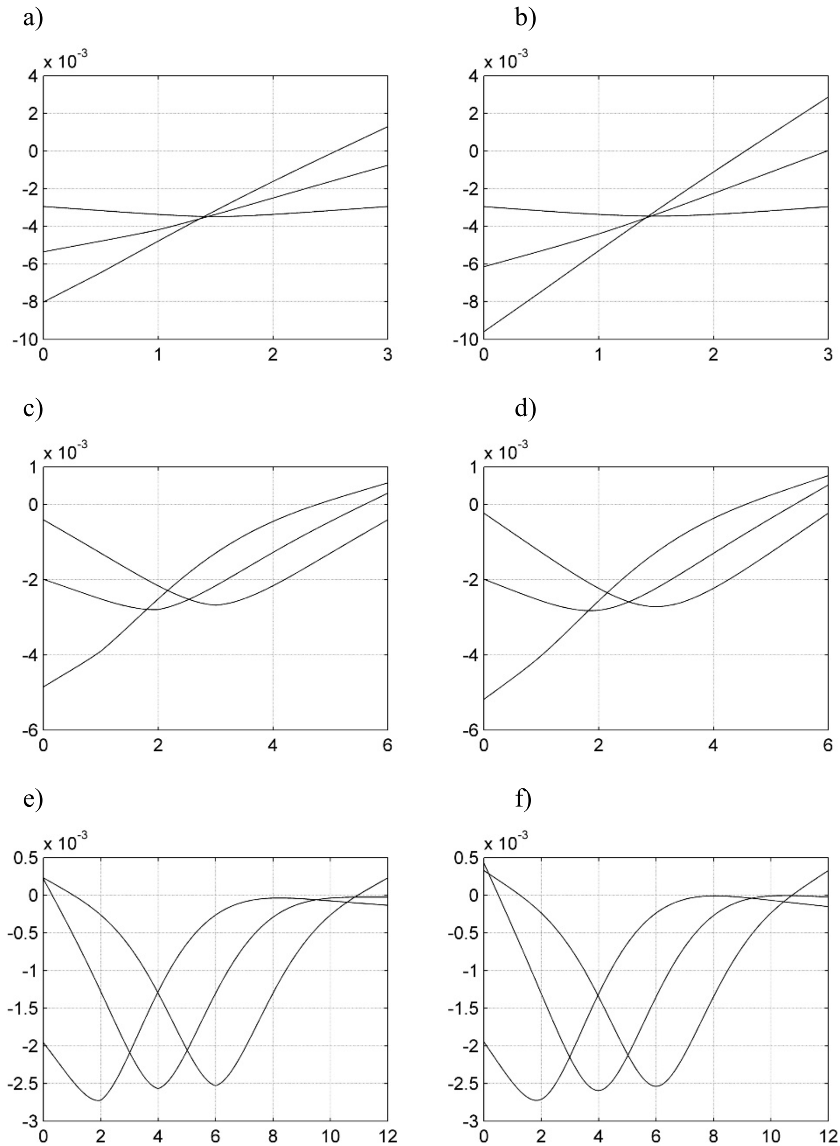


Fig. 14.7. Displacement function for the beam for various positions of concentrated load, length of the beam L = 3 m (a, b), 6 m (c, d), 12 m (e, f) computational model of the beam - variant T2 (a, c, e) and variant BE1 (b, d, f).

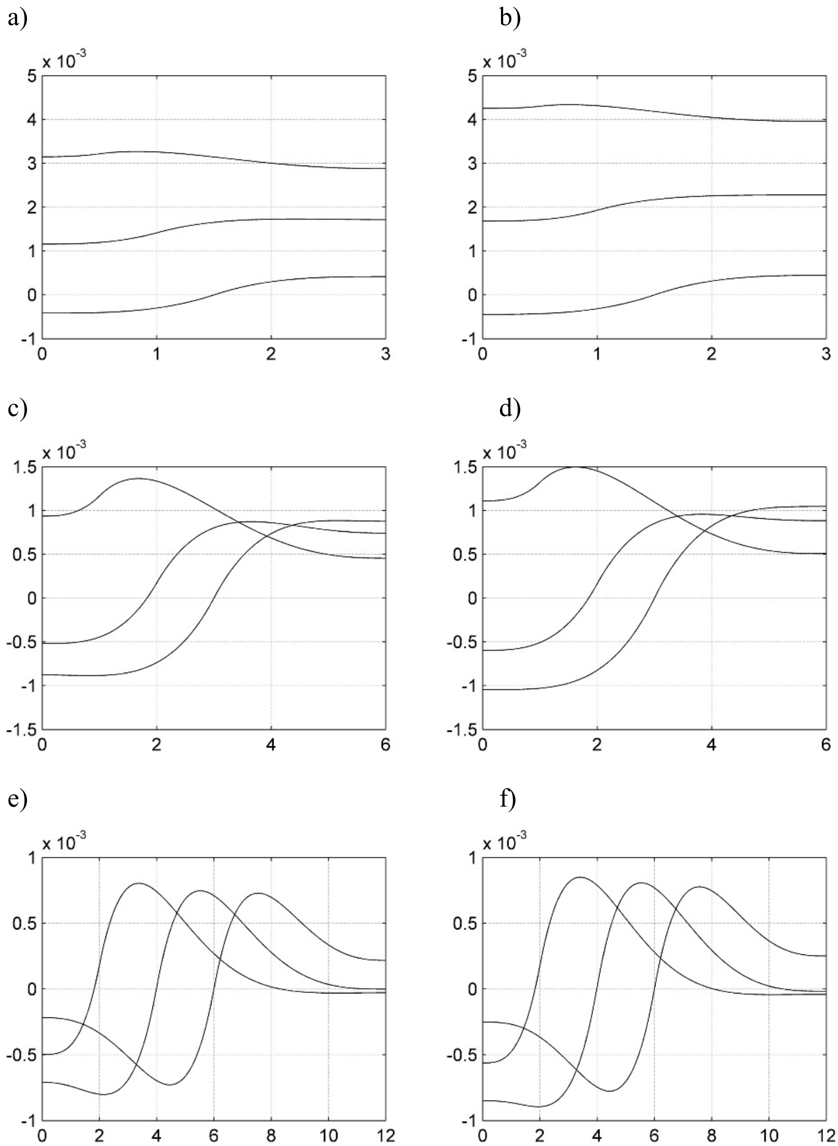


Fig. 14.8. Rotation angle for the beam for various positions of concentrated load, length of the beam L = 3 m (a, b), 6 m (c, d), 12 m (e, f) computational model of the beam - variant T2 (a, c, e) and variant BE1 (b, d, f).

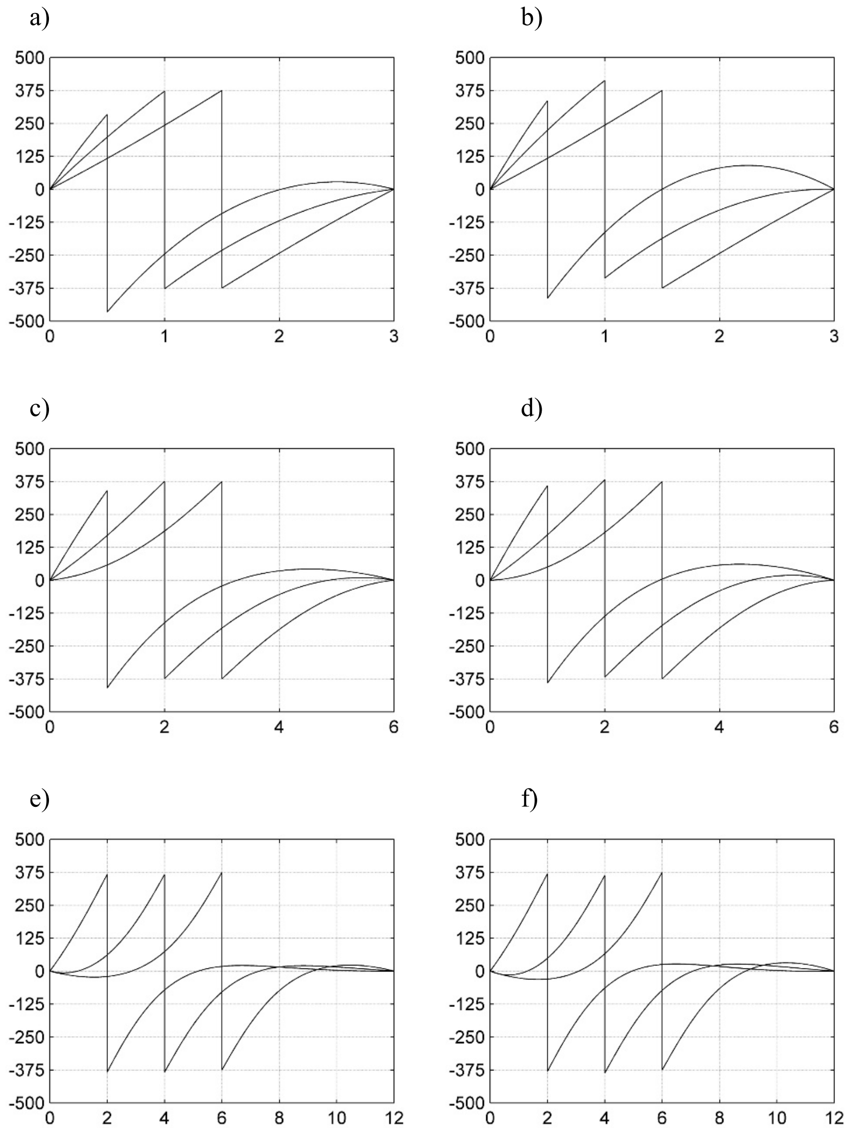


Fig. 14.9. Transversal force function for the beam for various positions of concentrated load, length of the beam L = 3 m (a, b), 6 m (c, d), 12 m (e, f) computational model of the beam - variant T2 (a, c, e) and variant BE1 (b, d, f).

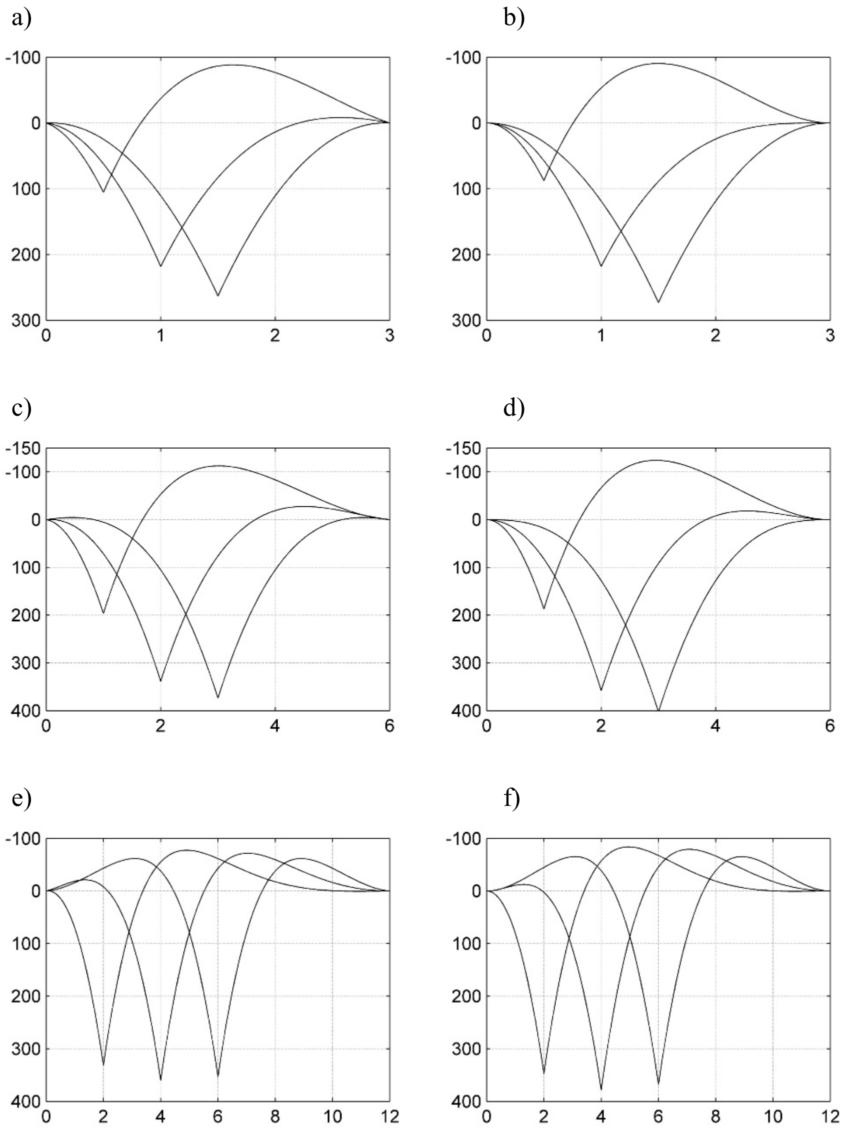


Fig. 14.10. Bending moment for the beam for various positions of concentrated load, length of the beam L = 3 m (a, b), 6 m (c, d), 12 m (e, f) computational model of the beam - variant T2 (a, c, e) and variant BE1 (b, d, f).

In Figures 14.7 and 14.8 considerable differences are visible for displacements and rotation angles for beams of 3 m length, loaded with forces P_1 or P_2. Differences are diminishing when the length grows (6 or 12 m) or we have the symmetric variant of load with force P_3. The diagrams of internal forces given in Figures 14.9 and 14.10 show the bigger differences for transversal forces than for bendig moments. The biggest differences are observed for short beams (L = 3 m, L/h = 5) loaded with forces P_1 or P_2.

14.6. Conclusions

Presented approach concerning analysis of rod structures with shear effects, interacting or not with two-parameter elastic foundation constitutes consistent and precise solution of the problem. It is characterized by an universal and simple approach in problem description, allowing for analysis of arbitrary particular cases, concerning various variants of analysis – taking into account or not the shear effects ($\eta = 0$), interaction with two- or one-parameter elastic foundation ($k_1 = 0$) and considering various loads: concentrated or distributed. This approach can be used for rods with consistent sections, three-layer sections or elements with truss sections. The presented equations can be applied in matrix version of displacement method or Finite Element Method.

Due to assumed very often high economics of design solutions, we should not be content with traditional models ignoring shear effects. This will protect against incorrect assessment of the situation.

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