KOSZALIN UNIVERSITY OF TECHNOLOGY

RESEARCH AND MODELLING IN CIVIL ENGINEERING 2019

Edited by
Jacek Katzer, Krzysztof Cichocki and Jacek Domski

KOSZALIN 2019

MONOGRAPH NO 366
FACULTY OF CIVIL ENGINEERING,
ENVIRONMENTAL AND GEODETIC SCIENCES

ISSN 0239-7129
ISBN 978-83-7365-525-6

Chairman of the University Editorial Board
Zbigniew Danielewicz

Editors
Jacek Katzer, Krzysztof Cichocki and Jacek Domski
Koszalin University of Technology, Poland

Reviewers
Jacek Gołaszewski – Silesian University of Technology, Poland
Wojciech Sumelka – Poznań University of Technology, Poland

Technical editor
Czesław Suchocki

Website editor
Mariusz Ruchwa

Linguistic consultations
Ewa Sokołowska-Katzer

Typesetting
Czesław Suchocki

Cover design
Tadeusz Walczak

www.cecem.eu

© Copyright by Koszalin University of Technology Publishing House
Koszalin 2019

This work is licensed under a Creative Commons Attribution-NonCommercial-ShareAlike 4.0 International License.

KOSZALIN UNIVERSITY OF TECHNOLOGY PUBLISHING HOUSE
75-620 Koszalin, Racławicka 15-17, Poland

Koszalin 2019, 1st edition, publisher's sheet 8,23, circulation 120 copies
Printing: INTRO-DRUK, Koszalin, Poland

Table of contents

1. Thermally loaded concrete specimens: fractal dimensions of fracture surface and mechanical fracture parameters 7

2. Contribution to comparison of methods for the investigation of chloride ingress related resistance 31

3. Requirements for BIM model 41

4. Comparative study of Building Law in Poland and Russian Federation......55

5. Numerical analysis of a semi-circular disc with an angled crack loaded in mixed-mode 71

6. Methodology for static analysis of pre-stressed plane cable systems 85

7. A UAV-based 3D model for building condition monitoring 107

8. Creating a web application to calculate predicted exploitative resistance to frost 121

9. The effect of steel fibres on selected properties of new generation of concrete 133

Research and modelling in civil engineering 2019
www.cecem.eu/monographs/

ISSN 0239-7129
ISBN 978-83-7365-525-6 pp. 85-106

6. Methodology for static analysis of pre-stressed plane cable systems

Dimitar Mitashev[1], Svetlana Lilkova-Markova[2]

[1]University of Architecture, Civil Engineering and Geodesy, Faculty of Hydraulic Engineering, Department of Technical Mechanics, Sofia, Bulgaria, mitashev@abv.bg,

orcid.org/0000-0001-6494-1879

[2]University of Architecture, Civil Engineering and Geodesy, Faculty of Hydraulic Engineering, Department of Technical Mechanics, Sofia, Bulgaria, lilkovasvetlana@gmail.com,

orcid.org/0000-0003-0582-8176

Abstract: Cable systems are an important structural type of systems. They are elegant, vivid and competitive and they always lead structures „towards the limit". The article presents the theoretical model, asumptions and the main elements' static analysis of one of the major types of cable systems – the "two-cables" plane systems. The article presents a description of the geometry and the elements of these systems, their static analysis and a somewhat simplified methodology for analysis with the use of conventional and not that complex softwares. The use of "main unknowns" in the methodology allows for an analytical derivation and expression of all system's state parameters - geometry, axial force in the main cables, tie forces etc. Some general conclusions about the influence of the pre-stress magnitude on displacements are made, also. The definition of the governing equations goes along with an analysis for unique solution for the systems. The commonly complex non-linear static analysis goes smoothly and is hidden behind a single governing equations system.

Keywords: cable, static, pre-stressed, non-linear

6.1. Theoretical model – general description and assumptions

Two-cable plane truss systems are composed from two types of elements: main cables and ties. The cases presented here follow the main ideas in (Mitashev, D., 2016) and account strictly for vertical ties. If the ties are inclined, the system gets more rigid, but at the end the inclination presents some difficulties only from mathematical point of view (for correct description of the system's geometry).

The main cables limit the system from above and below and the vertical ties are the connection between these main cables.

The main cables are two, one of them is convex down and the other one – up. They are usually called "bearing" and "prestressed". The structural function of the main cables depends on load position, pre-stress method etc.

The down-convex cable is assumed to be called "top cable", and the up-convex cable – "bottom cable".

The vertical ties connect the main cables. They can be designed as rods or/and cables. Of course, they can be designed as rods in any cases. Whether they can be designed as cables depends on the system configuration with account for all possible load cases.

The basic configuration of the system for which the theoretical study is made is as shown on Fig.6.1. From the obtained expressions, with accepting some variables as algebraic, the expressions for the other possible system configurations can be easily obtained. As an example, the distances l_i and h in (6.3) and (6.4) can get negative values.

There are three major system configurations, shown on Fig.6.1., Fig.6.2. and Fig.6.3.

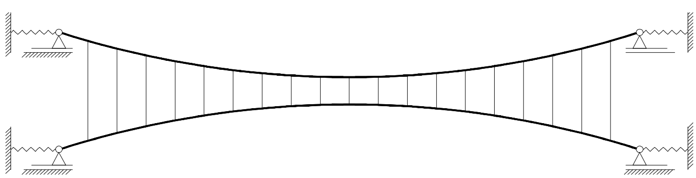

Fig. 6.1. System with vertical ties under tension

The system configuration shown on Fig.6.1. allows all vertical ties to be designed as cables.

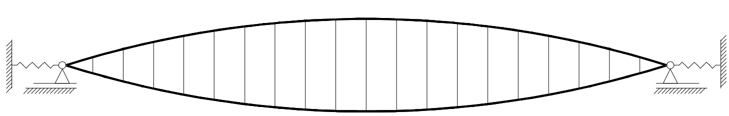

Fig. 6.2. System with vertical ties under compression

The system configuration shown on Fig.6.2. demands all vertical ties to be designed as rods because the inner system is under compression.

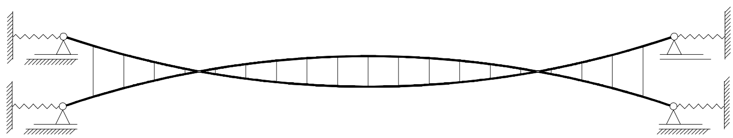

Fig. 6.3. System with vertical ties under tension & compression

The system configuration shown on Fig.6.3., allows the verticals between the supports and the inflection points of the main cables to be designed as cables, and demands for the rest of the ties to be designed as rods. That system is somewhat a mixed case of the other two and combines both their advantages and disadvantages.

General assumptions:

1. The main cables have constant cross section. The cross sections may be different for each of them independently.
2. The supports for each main cable are on the same level.
3. Vertical ties are accepted to be not-deformable. That is a common acceptance. It is assumed that vertical deformability has an insignificant influence on forces and displacements of the main cables. Some clarification on that issue is given in paragraph 6.3.
4. Forces are acting only in nodes. They are concentrated and vertical.
5. Main cables are with small sag (in the spirit of technical cable theory).
6. The structural material used behaves in accordance with Hooke's law. Strains are as for common structural materials i.e. small.
7. Due to the small sag of the main cables and their small deformability the loads are assumed to be acting along constant vertical lines.
8. Vertical ties of the system remain on these same vertical lines.
9. Elastic displacements of the supports do not influence the ratio between the lengths of the separate spans – that ratio remains constant while the total span of the system becomes smaller or larger.
10. When calculating the length of the main cables, because of their small sag, the square root under the integral in (6.21) is substituted with two members of the power series expansion.

11. Due to the main cables' small sag, normal forces in them are substituted with their horizontal projections for the small dimensions (variables) – elasticlengthening/shortening, temperature lengthening/shortening, support displacements. For that same reason instead of using the length of the cables the total span length is used.

It is seen from the assumptions listed above, that for the main cables the technical cable theory is accepted with two additional assumptions – that the verticals are non-deformable they remain constant on a vertical line.

6.2. Element numbering

The vertical lines that cross the supports and the verticals of the system are number from left to right with $0,1,........,n$. The separate spans between them have numbers $1,2,........,n$. That results in separate spans and boundary vertical line from their right side having the same numbers. The vertical ties have the same number as the vertical line they stand on i.e. $1,2,........,n-1$. All dimensions (variables) etc. corresponding to the top cable have an index of 1, and these for the bottom cable – 2.

Each of the main cables might be assumed as composed from a number of separate straight cables. They resemble the part of the main cable that is situated between two vertical lines. These separate cables and all dimensions (variables) etc. corresponding to them have a two-figure numbering: the number of the main cable and the number of the separate span they are in.

Nodes also have two-figure numbers – the number of the main cable and the number of the vertical line.

All element numbering and some other designations are given on Fig.6.4.

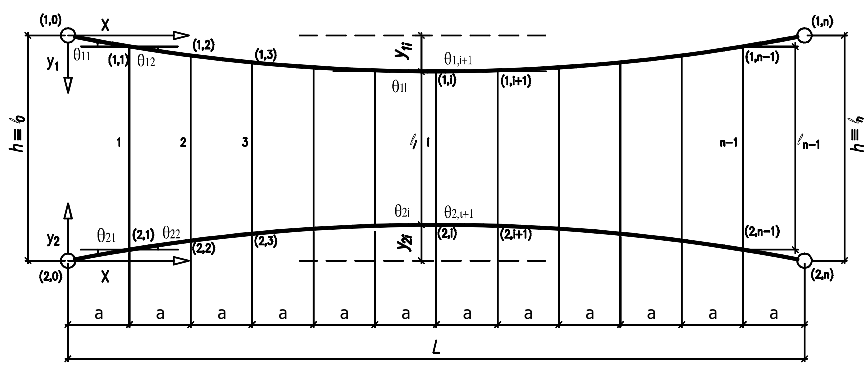

Fig. 6.4. System scheme and element numbering

6.3. Main unknowns

It is assumed that for all load states of the system, the external loads like self-load, live-load, temperature etc. are previously known, exactly as for the geometric and physical characteristics of the cables. Unlike for the conventional truss system, knowing all these parameters here is not enough for unambiguousdefinition of the cable-truss system variables: cable forces, nodal displacements etc.

Cable-truss systems need additional variables to be defined and their type depends on specific problem solved. It is best for these variables to be defined in a way that allows a following simple definition and calculation of all other system variables, like forces, displacements etc. These additional variables are called "main unknowns". For the systems that are analyzed here, the most suitable main unknowns are the thrusts in the main cables H_1 *and* H_2.

The thrusts represent the horizontal projections of the normal forces in the main cables. According to the previously assumptions made, the thrusts H_1 *and* H_2 are constant in all sections of the corresponding main cable. It can be firstly assumed that these two thrusts are the horizontal projections of the reactions in the left supports.

The thrusts H_1 *and* H_2 give us the opportunity to define all other variables of the system. The expressions shown below give the ability to solve problems in which the thrusts are previously given, as well as problems where other variables are previously known. In that second case, we should use some of the expressions in order to define (calculate) the main unknowns.

6.4. System variables expressed by the main unknowns

With the assumptions previously made, each of the cables turns out to be loaded with only two kinds of nodal forces : external vertical forces applied in the nodes - $F_{\alpha i}$ and forces in the vertical ties S_i $(\alpha = 1,2 \ ; \ i = 1,2,....,n-1)$. The forces $F_{\alpha i}$ are taken as positive when acting downwards. Forces in the vertical ties S_i are assumed to be positive if they are acting downwards for the top main cable and upwards for the bottom main cable. (Fig.6.5).

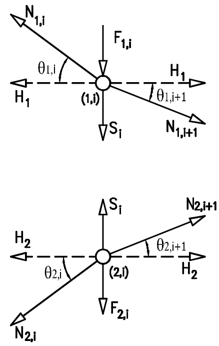

Fig. 6.5. Nodal external forces, main cable forces and forces in the vertical ties

According to the assumptions, the main cable ordinates y_{1i} *and* y_{2i} are calculated with the expressions

$$y_1(x)=\frac{M_1^{(SUB)}(x)}{H_1}$$
$$y_2(x)=\frac{M_2^{(SUB)}(x)}{H_2} \qquad (6.1)$$

Positive ordinates for the top cable are measured downwards from the line, connecting the supports of the top cable.

Positive ordinates for the bottom cable are measured upwards from the line, connecting the supports of the bottom cable.

$M_1^{(SUB)}(x)$ *and* $M_2^{(SUB)}(x)$ are the bending moments in the so called "substituting beam".That beam is simply supported and has the same span and loading as the corresponding main cable. Bending moments are assumed to be positive when causing increase in length of the bottom fibers of the substituting beam. In the expression for y_2 is taken into account that these ordinates are taken as positive when measured upwards. The abscissas x are measured to the right from the vertical line crossing the left supports.

As long as the superposition principle is valid, the expressions (6.1) can be presented also as

$$y_1(x)=\frac{M_1(x)+M_S(x)}{H_1}$$
$$y_2(x)=\frac{-M_2(x)+M_S(x)}{H_2} \qquad (6.2)$$

where:

$M_1(x)$ is the bending moment caused by the nodal forces F_{1i} acting on the top main cable

$M_2(x)$ is the bending moment caused by the nodal forces F_{2i} acting on the bottom main cable with $(i = 1, 2, ..., n-1)$.

$M_S(x)$ is the bending moment caused by the vertical forces S_i acting on the top main cable, again with $(i = 1, ..., n-1)$. In the expression for y_2 it is taken into account that these ordinates are taken as positive when measured upwards from the line connecting the supports. It is obvious that loads and forces acting on the bottom main cable shall have as a result bending moments with equal values and opposite signs.

Bending moments $M_1(x)$ *and* $M_2(x)$ are practically known, because they are easily found by the loading given. The question about the bending moment $M_S(x)$ though is somewhat different. That bending moment is unknown because the forces S_i in the vertical ties are also unknown. In order to find the unknown forces S_i we need some additional relations.

One of the assumptions made previously was for non-deformability of the vertical ties. In all cases when the vertical ties work in compression and are designed as rods, their cross sections are usually significantly greater than needed for strength purposes – that comes as a result of their stability control check. So their linear elastic deformations become naturally small. The influence of the vertical ties is not substantial even in cases when they work in tension. That is because small elastic deformations of the vertical ties have as a result a significantly smaller change in the lengths of the main cables.

When the assumption for non-deformability of the vertical ties is taken into consideration, then between the node ordinates y_{1i} and y_{2i} of the main cables exists the following relation:

$$y_{1i} + y_{2i} + l_i = h \quad , \quad (i = 1, ..., n-1) \tag{6.3}$$

where:

- l_i is the length of the vertical tie i. If the upper node of the vertical tie belongs to the top main cable, the length is taken as positive, and if it belongs to the bottom main cable – as negative.
- h is the distance between the lines connecting the supports of the top and bottom main cables respectively. That variable is taken as negative in case the top

main cables supports are situated entirely under the bottom main cable supports (a modified system configuration of that shown on Fig.6.2., when the main cables are with supports on different levels).

The number of the equations (6.2) is equal to the number of the unknown nodal ordinates and the number of equations (6.3) – equal to the number of the unknown forces. Therefore equations (6.2) and (6.3) allow the unknown variables to be expressed by the main unknowns.

We shall define a polygonal function $\varphi(x)$. The functions' values for the vertical lines with abscissas x_i are:

$$\varphi_i = \varphi(x_i) = h - l_i \qquad , \qquad (i = 0,1,....,n) \tag{6.4}$$

The lengths of the start and end vertical lines l_0 *and* l_n are in fact the vertical distance between the supports.

The function $\varphi(x)$ is linear between the nodes of the system.

Then from (6.3) and (3.4) it is easily seen that:

$$y_{1i} + y_{2i} = \varphi_i \qquad , \qquad (i = 0,1,....,n) \tag{6.5}$$

One of the assumptions of the theoretical model is that on the main cables are acting only concentrated forces that are applied in the nodes and are with vertical direction. So we conclude that bending moments between the nodes for the "substituting beam" also change linearly. Then it follows from the relations (6.2) that the ordinates $y_1(x)$ *and* $y_2(x)$ of the main cables also change linearly.

As a conclusion, the relation (6.5) shall be valid for each value of φ_i, and so:

$$y_1(x) + y_2(x) = \varphi(x) \tag{6.6}$$

The function $\varphi(x)$ is ambiguously defined with its nodal values. These values are obtained from (6.4), while at the same time h and l_i do not change even if the loads change. Then according to the non-deformability assumption it follows that the function $\varphi(x)$ does not depend on the loads and remains with constant values in each load state of the cable-truss system.

The bending moment $M_S(x)$ can be expressed from (6.2) as:

$$\begin{aligned} M_S(x) &= -M_1(x) + H_1 y_1(x) \\ M_S(x) &= M_2(x) + H_2 y_2(x) \end{aligned} \tag{6.7}$$

After considering the right sides of (6.7) being equal and some following transformations, we get the relation:

$$M_1(x)+M_2(x)=H_1y_1(x)-H_2y_2(x) \tag{6.8}$$

Expression (6.8), except the main unknowns H_1 *and* H_2, contains also the ordinates $y_1(x)$ *and* $y_2(x)$ in linear order. That gives us the opportunity using (6.6) and (6.8) to express these ordinates by the main unknowns:

$$\begin{aligned} y_1(x)&=\frac{M(x)+H_2\varphi(x)}{H_1+H_2}=\frac{M(x)}{H_1+H_2}+\frac{H_2\varphi(x)}{H_1+H_2} \\ y_2(x)&=\frac{-M(x)+H_1\varphi(x)}{H_1+H_2}=-\frac{M(x)}{H_1+H_2}+\frac{H_1\varphi(x)}{H_1+H_2} \end{aligned} \tag{6.9}$$

where it is substituted:

$$M(x)=M_1(x)+M_2(x) \tag{6.10}$$

$M(x)$ in equation (6.10) is the bending moment in the substitution simply supported beam caused by the simultaneous application of all nodal loads on both the top and the bottom main cables. It is obvious that $M(x)$ can be accepted as previously known because it is dependable only from the nodal loads.

In expressions (6.9) loading is represented only by the summary bending moment $M(x)$. On the other hand, the bending moment $M(x)$ does not change if some of the loads acting on one of the main cables is transferred to the other main cable (but on to the same vertical line) i.e. if some of the concentrated forces are transferred vertically from one to the other main cable.

Furthermore, the main unknowns H_1 *and* H_2 and the ordinates y_1 *and* y_2 remain also unchanged. The reason for that is the previously made assumption for the vertical ties to be non-deformable. The ideally rigid body allows loads to be transferred in that manner while the equilibrium state of the body remains unchanged. But it is important to say, that the transition mentioned above changes the internal forces of the system.

The following expressions show that forces in the vertical ties are dependent of the application place of the loads – whether they are applied over the top or over the bottom cable.

With substitution of (6.9) in (6.7) we obtain the bending moment $M_S(x)$, that is a result of the forces in the vertical ties, now expressed by the main unknowns:

$$M_S(x)=\frac{H_1M_2(x)-H_2M_1(x)+H_1H_2\varphi(x)}{H_1+H_2} \tag{6.11}$$

Expression (6.11) does not include the summary bending moment $M(x)$, but the bending moments $M_1(x)$ *and* $M_2(x)$ instead. The reason is that forces in the vertical ties change when loads are transferred, even though the equilibrium of the system remains unchanged. So $M_S(x)$ is changing dependently of $M_1(x)$ *and* $M_2(x)$, which on the other hand are dependable of the load application place (over the top or the bottom cable), and at the same time the summary bending moment $M(x)$ remains constant.

After differentiating of expressions (6.9) and using the relation $Q(x)=\dfrac{dM(x)}{dx}$ we obtain expressions for the inclination of the simple cables in function of the main unknowns:

$$y_1^{'} = tg\theta_1 = \frac{Q(x) + H_2\varphi^{'}(x)}{H_1 + H_2}$$
$$y_2^{'} = tg\theta_2 = \frac{-Q(x) + H_1\varphi^{'}(x)}{H_1 + H_2} \qquad (6.12)$$

where:

θ_1 *and* θ_2 are the angles between the tangent to the simple cables and a horizontal axis pointing to the right. The direction for measuring the angles θ_1 *and* θ_2 is from the tangent towards the cable.

If we add and index i for each separate span, then (12) become:

$$y_{1i}^{'} = tg\theta_{1i} = \frac{Q_i(x) + H_2\varphi_i^{'}(x)}{H_1 + H_2}$$
$$y_{2i}^{'} = tg\theta_{2i} = \frac{-Q_i(x) + H_1\varphi_i^{'}(x)}{H_1 + H_2} \qquad (6.13)$$

By using the expressions (6.12) we can obtain the normal forces in the separate simple cables. Inclinations, shear forces and the derivative $\varphi^{'}(x)$ have constant values in the separate spans.

The normal forces in the separate simple cables are:

$$N_{1i} = \frac{H_1}{\cos\theta_{1i}}$$
$$N_{2i} = \frac{H_2}{\cos\theta_{2i}} \qquad (6.14)$$

The forces in the vertical ties can be presented as concentrated forces, acting upon the substituting simply supported beam. Differentiating of (6.11) and taking into account that bending moment derivatives are constant between the vertical lines i.e in the separate spans, we get:

$$Q_{Si} = \frac{H_1 Q_{2i} - H_2 Q_{1i} + H_1 H_2 \varphi'(x)}{H_1 + H_2} \qquad ,(i = 1,....,n) \qquad (6.15)$$

Now we separate a part of the substituting beam with sections placed on an infinite small distances to the left and to the right from a vertical tie (Fig.6.6). We must also remember that if in the node there is a concentrated load present, that load has already been taken into account in the shear force value.

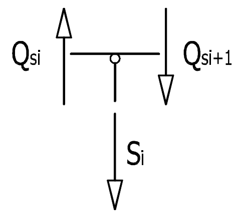

Fig. 6.6. Internal forces in vertical ties

If we write the equation for equilibrium $\Sigma V_i = 0$ we get the relation:

$$S_i = Q_{Si} - Q_{Si+1}. \qquad (6.16)$$

After expressing of Q_{Si} *and* Q_{Si+1} from (6.15), substituting in (6.16) and further transformations, the result for S_i is :

$$S_i = \frac{H_1(Q_{2i} - Q_{2i+1}) - H_2(Q_{1i} - Q_{1i+1}) + H_1 H_2(\varphi_i' - \varphi_{i+1}')}{H_1 + H_2} \qquad (6.17)$$

The same way as for (6.16) we obtain:

$$Q_{1i} - Q_{1i+1} = F_{1i} \quad u \quad Q_{2i} - Q_{2i+1} = F_{2i} \qquad (6.18)$$

Then (6.17) becomes:

$$S_i = \frac{H_1 F_{2i} - H_2 F_{1i} + H_1 H_2(\varphi_i' - \varphi_{i+1}')}{H_1 + H_2} \qquad (6.19)$$

In (6.19) forces in the vertical ties S_i are expressed by the main unknowns.

These forces can also be obtain by using the equation for equilibrium for the upper and/or lower node of the vertical tie (Fig.6.5) :

$$\begin{aligned} S_i &= -F_{1i} + H_1(y_{1i}' - y_{1i+1}') \\ S_i &= F_{2i} + H_2(y_{2i}' - y_{2i+1}') \end{aligned} \qquad (6.20)$$

In case of stationary supports without displacement, the lengths $\overline{L_1}$ *and* $\overline{L_2}$ of the prestressed main cables are calculated using the following expressions:

$$\overline{L_1} = \int_0^l \sqrt{1+y_1'^2}\,dx \approx \int_0^l (1+\frac{1}{2}y_{1i}'^2)\,dx + \frac{1}{2}\int_0^l y_{1i}'^2\,dx$$
$$\overline{L_2} = \int_0^l \sqrt{1+y_2'^2}\,dx \approx \int_0^l (1+\frac{1}{2}y_{2i}'^2)\,dx + \frac{1}{2}\int_0^l y_{2i}'^2\,dx \qquad (6.21)$$

We substitute (6.13) in (6.21) and after some simple transformations we get:

$$\overline{L_1} = l + \frac{1}{2}\frac{D+2BH_2+CH_2^2}{(H_1+H_2)^2}$$
$$\overline{L_2} = l + \frac{1}{2}\frac{D-2BH_1+CH_1^2}{(H_1+H_2)^2} \qquad (6.22)$$

where:

$$D = \int_0^l Q^2 dx = \sum_{i=1}^n Q_i^2 a_i \;\; ; \;\; B = \int_0^l Q\psi\,dx = \sum_{i=1}^n Q_i \psi_i a_i$$
$$C = \int_0^l \psi^2 dx = \sum_{i=1}^n \psi_i^2 a_i \qquad (6.23)$$

The expressions (6.23) define the geometry and the loading of the cable-truss system. The geometry is defined by $\psi = \varphi'(x)$ and the loading influence is taken into account by the shear force $Q(x)$ in the substituting beam. In (6.23) for simplicity the substitution is:

$$\psi(x) = \varphi'(x) \;\; , \;\; \psi_i = \varphi_i' \qquad (i = 1, ..., n) \qquad (6.24)$$

where:

φ_i' is the derivative of φ in separate span i and a_i is the span distance between the vertical ties of the system.

All expressions up to that point are made with the assumption of the supports not allowing displacement.

For the common structural systems support displacements turn out to have small values. That is the reason for them to be ignored in all expressions up to here, except in these for calculating the length of the cables. As we can see, expressions (6.22) consider these variables. Support displacements are small compared to the cable length. Expressions (6.22) give us the differences between the cable lengths and the distance between the supports (the total span distance). These differences are small quantities.

Support displacements change the distances between the vertical lines and, and as a result, the expressions (6.25). These second members of (6.22) are small compared with the first – the total span distances. The increase of $\overline{d_1}$ *and* $\overline{d_2}$ are quantities of a higher order than the increase of the first member (the total span distances) and therefore they can be ignored.

If we take into account for supports' displacements, the lengths of the prestressed cables are presented as follows:

$$\overline{L_1} = l - \Delta_{\text{sup},1} + \frac{1}{2}\frac{D + 2BH_2 + CH_2^2}{(H_1 + H_2)^2}$$
$$\overline{L_2} = l - \Delta_{\text{sup},2} + \frac{1}{2}\frac{D - 2BH_1 + CH_1^2}{(H_1 + H_2)^2} \quad (6.26)$$

$\Delta_{\text{sup},1}$ and $\Delta_{\text{sup},2}$ are the mutual horizontal support displacements for the corresponding cable.

In this article we assume linearly-elastic supports that allow only horizontal displacements.

In that case we have:

$$\delta_{\text{sup},1} = \delta_{11}H_1 + \delta_{12}H_2$$
$$\delta_{\text{sup},2} = \delta_{21}H_1 + \delta_{22}H_2 \quad (6.27)$$

where:

- δ_{11} is the mutual displacement of the top cable supports caused by thrusts $H_1 = 1 \; u \; H_2 = 0$
- δ_{22} is the mutual displacement of the bottom cable supports caused by thrusts $H_1 = 0 \; u \; H_2 = 1$
- δ_{12} is the mutual displacement of the top cable supports caused by thrusts $H_1 = 0 \; u \; H_2 = 1$
- δ_{21} is the mutual displacement of the bottom cable supports caused by thrusts $H_1 = 1 \; u \; H_2 = 0$

The Maxwell-Betti theorem states that:

$$\delta_{21} = \delta_{12} \quad (6.28)$$

From the cable lengths $\overline{L_1}$ *and* $\overline{L_2}$ when they are prestressed we can easily obtain their lengths L_1 *and* L_2 when they are not prestressed. Elastic changes in length are Δ_{e1} *and* Δ_{e2}. For calculation of L_1 *and* L_2 it is enough to subtract the elastic changes in length Δ_{e1} *and* Δ_{e2} from the prestressed cables' lengths $\overline{L_1}$ *and* $\overline{L_2}$.
The mutual horizontal displacements of the supports $\Delta_{\sup,1}$ and $\Delta_{\sup,2}$ can be written as:

$$\begin{aligned} \Delta_{\sup,1} &= \Delta_{e1} + \delta_{\sup,1} \\ \Delta_{\sup,2} &= \Delta_{e2} + \delta_{\sup,2} \end{aligned} \tag{6.29}$$

The exact expressions for the elastic changes in lengths Δ_{e1} *and* Δ_{e2} are:

$$\begin{aligned} \Delta_{e1} &= \int_{x=0}^{x=l} \frac{N_1}{E_1 A_1} ds = \frac{H_1}{E_1 A_1} \int_0^l (1 + \frac{1}{2} y_1'^2) dx \\ \Delta_{e2} &= \int_{x=0}^{x=l} \frac{N_2}{E_2 A_2} ds = \frac{H_2}{E_2 A_2} \int_0^l (1 + \frac{1}{2} y_2'^2) dx \end{aligned} \tag{6.30}$$

where:
- N_1 *and* N_2 are the normal forces in the main cables;
- E_1 *and* E_2 are the elastic material modulus of the main cables;
- A_1 *and* A_2 are the cross-sectional areas of the main cables.

The squares of the derivatives in (6.30) are small compared to the number 1 and they can therefore be ignored. We obtain the following simple expressions:

$$\begin{aligned} \Delta_{e1} &= \frac{H_1 l}{E_1 A_1} \\ \Delta_{e2} &= \frac{H_2 l}{E_2 A_2} \end{aligned} . \tag{6.31}$$

After substituting (6.31) in (6.29) and taking into account (6.27), the lengths of the prestressed main cables expressed by (26) transform in:

$$\begin{aligned} \overline{L_1} &= l - \delta_{11} H_1 - \delta_{12} H_2 - \frac{H_1 l}{E_1 A_1} + \frac{1}{2} \frac{D + 2BH_2 + CH_2^2}{(H_1 + H_2)^2} \\ \overline{L_2} &= l - \delta_{21} H_1 - \delta_{22} H_2 - \frac{H_2 l}{E_2 A_2} + \frac{1}{2} \frac{D - 2BH_1 + CH_1^2}{(H_1 + H_2)^2} \end{aligned} \tag{6.32}$$

On the right side of (6.32) there are variables from different order – the total span distance l is much greater compared to all other members.

We now introduce the variables Δ_1 *and* Δ_2 which are of the same order as the small members in (6.32).

$$\begin{aligned} \Delta_1 &= l - \overline{L_1} \\ \Delta_2 &= l - \overline{L_2} \end{aligned} . \tag{6.33}$$

It follows from (6.32) and (6.33) that:

$$\begin{aligned} \Delta_1 &= \delta_{11}H_1 + \delta_{12}H_2 + \frac{H_1 l}{E_1 A_1} - \frac{1}{2}\frac{D + 2BH_2 + CH_2^2}{(H_1 + H_2)^2} \\ \Delta_2 &= \delta_{21}H_1 + \delta_{22}H_2 + \frac{H_2 l}{E_2 A_2} - \frac{1}{2}\frac{D - 2BH_1 + CH_1^2}{(H_1 + H_2)^2} \end{aligned} \tag{6.34}$$

The newly introduced variables Δ_1 *and* Δ_2 have a very simple meaning. If they are positive/negative, they show how shorter/longer are going to be the non-prestressed cable lengths compared to the total span distance, i.e. how much the main cables shall need/lack" to be exactly equal to the total span distance.

It is assumed Δ_1 *and* Δ_2 to be called "insufficiencies" of the main cables for the given total span distance.

6.5. Governing equations for the main unknowns

If for any given load state the thrusts H_1 *and* H_2 are previously known, then all other variables of the system state can be easily found i.e. the system state is fully defined.

When the thrusts are known expressions (6.34) define the so called "insufficiencies". The case here is of known insufficiencies and solving the problem for obtaining the thrusts.

In that case expressions (6.34) become an equation system for obtaining the thrusts. The system can be written in the following form:

$$\begin{aligned} r_1(H_1, H_2) &= \delta_{11}H_1 + \delta_{12}H_2 + \frac{H_1 l}{E_1 A_1} - \frac{1}{2}\frac{D + 2BH_2 + CH_2^2}{(H_1 + H_2)^2} - \Delta_1 = 0 \\ r_2(H_1, H_2) &= \delta_{21}H_1 + \delta_{22}H_2 + \frac{H_2 l}{E_2 A_2} - \frac{1}{2}\frac{D - 2BH_1 + CH_1^2}{(H_1 + H_2)^2} - \Delta_2 = 0 \end{aligned} \tag{6.38}$$

The left sides of $r_1(H_1,H_2)$ *and* $r_2(H_1,H_2)$ in expressions (6.38) are algebraic expressions of the main unknowns H_1 *and* H_2.

6.6. Governing equations for the main unknowns

The general problem, the same as in the case of a single cable, can be formulated in the following manner:

One load state, called initial state is fully defined, that is the starting "known state" of the system. Then a change in loads and/or temperature occurs. The goal is to define the new state of the system. That new state is called final state or "wanted state".

All common geometrical and physical variables for the two states are previously known: cross-sectional areas, elasticity modulus, linear temperature coefficients, vertical ties lengths (and so the nodal values of the function $\varphi(x)$. These variables are exactly the same for the two states – the known "initial state" and the wanted "final state".

We are also aware of the loads and temperatures for both states. The "initial state" is fully defined because the thrusts H_{01} *and* H_{02} are previously known. The main unknowns for the wanted "final state" are the thrusts H_1 *and* H_2. Once defined, all other system variables for that state can be ambiguously calculated and so the state shall become fully defined.

All variables that are different for the two cases are named in the previously defined in paragraph 2 way – exactly the same for the "final state", and for the "initial state" - with adding the index "zero".

For composition of the governing equation of the problem we must first of all obtain the relation between the lengths of the two main cables. In general, they are different for the two states because of the temperature difference and the resulting elastic temperature deformations in the main cables:

$$\begin{aligned} L_1 &= L_{01} + \alpha_{t1}(t_1 - t_{01})l \\ L_2 &= L_{02} + \alpha_{t2}(t_2 - t_{02})l \end{aligned}, \qquad (6.39)$$

where:

α_{t1} *and* α_{t2} - linear temperature deformation coefficients for the main cables

$t_{01},\ t_{02},\ t_1$ *and* t_2 - temperatures of the main cables in the "initial" and "final" states.

When calculating the elastic temperature deformation (increase or decrease of length) with (6.39), instead of the main cable length the total span length is used. It is assumed that both the elastic temperature deformations and the difference between the main cable and total span lengths are variables of small order. It turns out to be more suitable to solve the problem working with the insufficiencies instead of the main cable lengths. That's because in the following expressions all parts are of the same order. Taking into consideration (6.33), (6.39) easily gives us the relation between the insufficiencies for the "initial" and the "final" states:

$$\begin{aligned} \Delta_1 &= \Delta_{01} - \alpha_{t1}(t_1 - t_{01})l \\ \Delta_2 &= \Delta_{02} - \alpha_{t2}(t_2 - t_{02})l \end{aligned} . \tag{6.40}$$

The insufficiencies for the "initial" state Δ_{01} *and* Δ_{02} are calculated using (6.34) and with the naming style defined for that state (34) becomes :

$$\begin{aligned} \Delta_{01} &= \delta_{11}H_{01} + \delta_{12}H_{02} + \frac{H_{01}l}{E_1A_1} - \frac{1}{2}\frac{D_0 + 2B_0H_{02} + C_0H_{02}^2}{(H_{01}+H_{02})^2} \\ \Delta_{02} &= \delta_{21}H_{01} + \delta_{22}H_{02} + \frac{H_{02}l}{E_2A_2} - \frac{1}{2}\frac{D_0 - 2B_0H_{01} + C_0H_{01}^2}{(H_{01}+H_{02})^2} \end{aligned} \tag{6.41}$$

The variables D_0, B_0 *and* C_0 here are calculated according to (6.23).

$$\begin{aligned} D_0 &= \int_0^l Q^2 dx = \sum_{i=1}^n Q_{0i}^2 a_i \quad ; \quad B_0 = \int_0^l Q\psi\, dx = \sum_{i=1}^n Q_{0i}\psi_i a_i \\ C_0 &= \int_0^l \psi^2 dx = \sum_{i=1}^n \psi_i^2 a_i \end{aligned} \tag{6.42}$$

After calculating the insufficiencies for the "initial" state with (6.41) and (6.42), the use of (6.40) gives as a result the insufficiencies for the "final" state.

That leads the problem to solving the governing equations system (6.38). The variables D, B *and* C in (6.38) are also calculated using (6.34) with the characteristics for the "final state". The variables C *and* C_0 turns out to be exactly the same for both cases because they do not include the loading parameters.

The governing equations system consists of two cubic algebraic equations with both main unknowns included. Solving the system usually demands the use of numerical methods.

Most importantly, the very significant question about the solution being unique, still stays opened.

One of the well-known solutions of this type of systems belongs to Dmitriev and Kasilov (Dmitriev L.G., Kasilov A.V., 1974) The thrust in the main cables is derived from an algebraic equation of the fifth power. The solution is somewhat easier than solving the (6.38) system, but it does not comply for the uniqueness demand!

6.7. Analysis of the governing equations

The governing equations (6.38) are in fact stationary conditions of a scalar function - $P = P(H_1, H_2)$.

In order for that to be correct it's needed and enough to have the following expressions proven true:

$$\frac{\partial r_1}{\partial H_2} = \frac{\partial r_2}{\partial H_1}$$

$$\delta_{12} - \frac{CH_1H_2 + B(H_1 - H_2) - D}{(H_1 + H_2)^3} = \delta_{21} - \frac{CH_1H_2 + B(H_1 - H_2) - D}{(H_1 + H_2)^3} \tag{6.43}$$

Expression (3.28) is assumed and the scalar function $P = P(H_1, H_2)$ can be delivered by an appropriate integration of the expressions r_1 and r_2 for the left parts of the equations.

We get the following:

$$P(H_1, H_2) = \frac{1}{2}\left[\begin{array}{l}(\delta_{11} + C_1)H_1^2 + (\delta_{22} + C_2)H_2^2 + \\ +2\delta_{12}H_1H_2 + \dfrac{D + B(H_2 - H_1) - CH_1H_2}{H_1 + H_2}\end{array}\right] + \overline{C} \tag{6.44}$$

In expression (6.44) $\overline{C}$ is an arbitrary constant. It's here assumed for it to be equal to zero. For simplicity it is assumed that:

$$C_1 = \frac{l}{E_1A_1} \qquad ; \qquad C_2 = \frac{l}{E_2A_2}\,. \tag{6.45}$$

C_1 *and* C_2 have the physical meaning of elongations for the specific cable from the thrust. Obviously, by differentiation of (6.44), for the scalar function $P = P(H_1, H_2)$ we get the left parts of the equations (6.38). That proves the statement, that the governing equations (6.38) are in fact stationary conditions of the function $P = P(H_1, H_2)$.

$$r_1(H_1,H_2)=\frac{\partial P}{\partial H_1}=(\delta_{11}+C_1)H_1+\delta_{12}H_2-\frac{1}{2}\frac{D+2BH_2+CH_2^2}{(H_1+H_2)^2}$$
$$r_2(H_1,H_2)=\frac{\partial P}{\partial H_2}=\delta_{21}H_1+(\delta_{22}+C_2)H_2-\frac{1}{2}\frac{D-2BH_1+CH_1^2}{(H_1+H_2)^2} \qquad (6.46)$$

The type and character of the function's stationarity (if the function has a maximum, a minimum or a stationary point) can be examined with a quadratic form of its second derivatives.

They are as follows:

$$\frac{\partial^2 P}{\partial H_1^2}=\delta_{11}+C_1+\frac{D+2BH_2+CH_2^2}{(H_1+H_2)^3} \qquad (6.47)$$

$$\frac{\partial^2 P}{\partial H_1 \partial H_2}=\delta_{12}+\frac{D-B(H_1-H_2)-CH_1H_2}{(H_1+H_2)^3} \qquad (6.48)$$

$$\frac{\partial^2 P}{\partial H_2^2}=\delta_{22}+C_2+\frac{D-2BH_1+CH_1^2}{(H_1+H_2)^3} \qquad (6.49)$$

By the above expressions we can see that their quadratic form can be presented as a sum of three quadratic forms with the following matrices:

$$T_1=\begin{vmatrix}\delta_{11} & \delta_{12}\\ \delta_{21} & \delta_{22}\end{vmatrix} \qquad T_2=\begin{vmatrix}C_1 & 0\\ 0 & C_2\end{vmatrix}$$
$$T_3=\frac{1}{(H_1+H_2)^3}\begin{vmatrix}D+2BH_2+CH_2^2 & D-B(H_1-H_2)-CH_1H_2\\ D-B(H_1-H_2)-CH_1H_2 & D-2BH_1+CH_1^2\end{vmatrix} \qquad (6.50)$$

The quadratic form of the first matrix resembles the doubled potential energy of the supports' deformations and is:

$$2U_{\text{supports}}=H_1{}^2\delta_{11}+2H_1H_2\delta_{12}+H_2{}^2\delta_{22}\geq 0 \qquad (6.51)$$

If the supports are horizontally elastic, it is positively determined, and if they're pinned – it's equal to zero.

The quadratic form of the second matrix resembles the doubled potential energy of the deformation of the cables. It is positively determined:

$$2U_{\substack{main\\cables}} = C_1 H_1^2 + C_2 H_2^2 > 0 \tag{6.52}$$

The quadratic form of the third matrix is examined with Sylvester's criterion. In order for it to be positively determined, it's enough the first major minors of each row of the matrix to be non-negative. In the case it means that the first element of the major diagonal and the determinant should be non-negative.

The first element of the major diagonal is non-negative, because according to (6.23) it can be represented as follows:

$$\int_0^l (Q + H_2\psi)^2 dx \geq 0 \tag{6.53}$$

That element shall be equal to zero only if $Q + H_2\psi \equiv 0$ and with following the boundary conditions, when

$$y_1 = \frac{M + H_2\varphi}{H_1 + H_2} \equiv 0 \tag{6.54}$$

The determinant is equal to

$$|T_3| = \frac{DC - B^2}{H_1 + H_2}. \tag{6.55}$$

By assuming (IV.23), for the numerator we get:

$$(DC - B^2) = \begin{vmatrix} D & B \\ B & C \end{vmatrix} = \begin{vmatrix} \int_0^l Q^2 dx & \int_0^l Q\psi dx \\ \int_0^l Q\psi dx & \int_0^l \psi^2 dx \end{vmatrix} \geq 0 \tag{6.56}$$

That's a Gramm's determinant and it's non-negative.

Because all three quadratic forms are proven to be positively determined, their sum is a positively determined quadratic form, also.

So, the scalar function $P(H_1, H_2)$ has a strict downwards convexity in the whole first quadrant, and its stationary point, if it exists, is a minimum.

The proven strict downwards convexity leads to the statement that the function has a single minimum, and by that – the solution of the governing equations' system delivers an unique solution.

6.8. Conclusions

The presented work shows that even complex geometrycally non-linear systems can be sometimes analyzed relatively easy. Article covers all analytical aspects for pre-stressed cable systems with vertical ties.

The methodics accounts for horizontally linear elastic supports and temperature difference.

Using a simple math software for implementing the methodics gives the oportunity to evade using more complex specialized softwares.

Ther most distinct difference with other known methods is the unique solution of the governing equations.

References

Mitashev, D. (2016) ‘Static and dynamic analysis of planar cable systems', PhD. Thesis, UACEG, Sofia, Bulgaria

Dmitriev L.G., Kasilov A.V., Cable-stayed roof structures (analysis and construction), 1974, Kiev